BAYESIAN APPROACHES
TO THE RELIABILITY ASSESSMENT OF EXISTING STRUCTURES

Dissertation

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by

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*) Either the German or the Italian form of the title may be used.

ABSTRACT

Due to environmental, economic and socio-political reasons, significance and field of application of existing structures reliability assessments' extend rapidly, also in view of the preservation of cultural heritage. Over recent decades, the Bayesian analysis has been increasingly recognized as an efficient procedure for the definition of the probability models for the random variables governing the reliability of the structure. It is especially applied in case of material mechanical parameters, where a prior is firstly defined according to engineering judgments and then updated with the result of material tests. However, if the choice of the prior distribution is wrong, the result of the analysis might be incorrect. Material tests are usually destructive, leading to a loss of the integrity of the structure and even of its cultural value, in case of heritage buildings. Therefore it is of paramount importance to propose procedures according to which more sound prior distributions are defined and non-destructive approach to the Bayesian analysis are promoted.

In this work, a general methodology that increases the robustness of the Bayesian analysis with respect to the prior distribution for the description of material mechanical random parameters is defined. The methodology focuses on the identification of homogeneous material classes considering certain mechanical properties, and the definition of the related statistical parameters, directly analysing the results of material acceptance tests. A practical application of this procedure is developed for the identification of concrete classes in Italy during the 1960s analyzing the results of compressive tests on standard cubes. On the other hand, it is shown by practical application on relevant case studies that a more realistic reliability evaluation can be obtained without touching the structure, refining the probability models of action or action effects. In the first case study (a concrete water tank from the 1960s) the Bayesian updating of the Finite Element model is carried out considering the results of static and dynamic tests. In the second case study (a masonry aqueduct built in the $16th$ century) the Bayesian updating of the wind speed is performed according to the results of field studies.

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Thesis outline

The thesis is subdivided in the following way: Chapter 1 defines the general framework and the objectives of this work; Chapter 2,3 and 4 review the state of the art respectively in the reliability assessment of existing buildings, in the Bayesian analysis and in the Bayesian approach to the stochastic inverse problem. Chapter 5, 6 and 7 illustrate new ideas and applications developed by the author, and thus they represent the most original part of the work. Chapter 8 draws the conclusions and the further steps of the research. In particular:

Chapter 1 identifies the motivation behind the research and the scope of the thesis, also presenting the practical applications that have been developed;

in **Chapter 2** the key elements involved in the reliability assessment of existing structures are presented. At first, a definition of reliability is given; in order to do this, the concepts of limit state, measure of failure and safety target are introduced. Paralleling to those technical issues, other aspects are tackled, especially the variety of existing structures and the motivation behind their assessment. This chapter intends to identify the main factors that should be considered approaching the reliability assessment of existing structures;

in **Chapter 3** the definition of the probability models for the basic random variables is considered. The Bayesian analysis is here proposed: a prior distribution representing the 'degree of belief' associated with the values of the parameter is updated with the results of experimental investigation applying Bayes' theorem. Special interest is devoted to the choice of the prior distribution and the computational aspects of the posterior distribution. The so-called posterior distribution is especially affected by the choice of the prior, and for this reason, several methods for the elicitation are considered, such as those based on subjective judgments, past data, and the concept of entropy. Besides classical approximations methods for the computation of the posterior distribution, attention is drawn on the use of conjugate prior distributions, that allow for an analytic solution, and the more general Markov Chain Monte Carlo methods;

Chapter 4 focuses on updating the probability distribution function of the parameters that govern the reliability of existing buildings using measurements of observable responses of the structure. Herein three stochastic inverse methods based on a functional approximation of the system response through general Polynomial Chaos Expansion theory have been presented: the Markov Chain Monte Carlo method, the Polynomial Chaos Expansion based Kalman Filter, and the parameter update with the Minimum Mean Squared Error estimator. It is shown in a toy example, how the functional approximation of the system response can be obtained. The potentialities of the proposed methods are also discussed considering the results of simple updating carried out on linear and non-linear analytical models;

in **Chapter 5** a novel method for the identification of material homogeneous population (or material classes) and their statistical parameters is presented. A fundamental step of the method is the cluster analysis based on Gaussian Mixture Model of secondary experimental test data. The proposed methodology is applied in order to investigate the concrete cubic strength of the Italian production during the 1960s. Applying this method it is also possible to identify the uncertainty affecting material classes' statistical parameters and to easily set up prior distributions that can be updated in a Bayesian analysis.

In **Chapter 6** the reliability assessment of a concrete water tank under seismic loads is presented. The study involves the definition and the analysis of the Finite Element model of the structure, which is characterised by random inputs. The response of the structure to static and dynamic loads has been measured in order to simultaneously update input random variables. Procedures based on a functional approximation of the system response with general Polynomial Chaos Expansion have been applied in order to solve the Bayesian Inverse Problem. The identified model has been used in order to perform the reliability assessment and define the seismic fragility curves for different intensity levels of the peak ground acceleration corresponding to the two extreme conditions of empty and full tank;

Chapter 7 presents an overall methodology for evaluating the safety of ancient buildings, that also addresses the collection of relevant information about the edifice. The procedure has been applied to a case study, a historic aqueduct in the nearby of Pisa in Italy, affected by settlements and out of plane overturning. The reliability under horizontal loads, namely seismic and wind actions, has been assessed, also considering the Bayesian updating of the wind speed.

Finally, in **Chapter 8** the results of the whole thesis are discussed and some suggestions and indications for future research are proposed.

List of Abbreviations

AIC Akaike Information Criterion

BIC Bayesian Information Criterion

CDF cumulative distribution function

COV coefficient of variation

DT destructive test

EM Expectation Maximization

EnKF Ensemble Kalman Filter

FEM finite element model

FORM First Order Reliability Method

GMM Gaussian mixture model

gPCE general Polynomial Chaos Expansion

i.i.d independent and identically distributed

KF Kalman Filter

MC Monte Carlo

MCMC Markov Chain Monte Carlo

MDT minor-destructive test

MH Metropolis-Hastings

MM mixture model

MMSE Minimum Mean Squared Error

NDT non-destructive test

NL-MMSE Non-Linear Minimum Mean Squared Error

PC Probability Component

PCE-KF Polynomial Chaos Expansion based Kalman Filter

pdf probability distribution function

PGA Peak Ground Acceleration

SLS serviceability limit state

SORM Second Order Reliability Method

ULS ultimate limit state

w/c water to cement ratio

Contents

Chapter 1

Introduction

In this chapter, the general framework and the main objectives of this work are defined, and the case studies are presented.

Contents

1.1 Introduction

1.1.1 General framework

The relevance of the reliability assessment of existing structures is rapidly increasing for environmental, economic and socio-political reasons also in view of the preservation of cultural heritage. Within this field, the Bayesian analysis has been recognized as a promising procedure for defining the probability models of the parameters governing the reliability of the structure.
The Bayesian approach is frequently applied for updating the probability distribution function of material mechanical parameters, where a prior distribution is firstly defined according to engineering judgments and then updated by means of experimental results, which can be obtained from non-destructive, minor-destructive or destructive tests (respectively NDTs, MDTs and DTs) usually performed in-situ. Anyhow, it must be underlined that DTs and even MDTs are often difficult to be performed, especially in case of heritage buildings, where they can lead to a loss of the integrity of the structure and even of its cultural value. At the same time, the right choice of the prior distributions is of crucial importance, since wrong assumptions could determine an incorrect conclusion.
Thus new procedures are sought, which foster NDTs and allow for the definition of more sound prior distributions.

1.1.2 Objectives

The objectives of the present Ph.D. thesis are twofold: on the one hand, a general methodology that increases the robustness of the Bayesian analysis with respect to the prior distribution is sought, paying particular attention to the definition of probability models for material mechanical parameters; on the other hand, the use of non-destructive approaches within the Bayesian analysis is promoted by practical applications to relevant case studies, where suitable approaches mainly focus on the updating of action or action effects through respectively direct realizations of the random variable or measurement of the system response to static and dynamic loads, also resorting to the Bayesian updating of the finite element model (FEM), so that reliability evaluation can be improved without touching the structure. The main findings of the thesis are then applied to two relevant case studies: in the first case study, the Medicean Aqueduct in Pisa, a masonry aqueduct built in the 16*th* century, the Bayesian updating of the wind speed is performed according to the results of field studies; in the second case study, a concrete water tank from the 1960s, the Bayesian updating of the FEM is carried out considering the results of static and dynamic tests.

Besides, an original and general method is proposed to define the prior distributions of mechanical parameters of the material, in order to increase the robustness of the Bayesian updating of these random variables, relying on test results obtained on structures similar to the one under consideration. The identification of the actual material properties is usually done considering reference homogeneous material classes, pertaining to similar structures, for which statistical parameters are known. Unfortunately, despite its apparent easiness, this approach is often frustrated by the unavailability of sound experimental test results: in effect, it is not obvious to identify among huge numbers of tested specimens those belonging to homogeneous populations. The proposed methodology allows recognizing homogeneous populations and their statistical parameters directly analysing the results of standard material tests, and it can be applied to every structural material, like steel, concrete, timber, masonry and so on. A relevant practical application of the procedure is finally developed for the identification of the statistical properties, and the coefficient of variation in particular, of Italian concrete classes during the 1960s, based on the results of compressive tests on standard cubes.

Chapter 2

Reliability assessment of existing structures

In this chapter, the basic understandings of the reliability assessment of existing structures are reviewed. Not only the technical aspects of the problem are treated, such as the concept of limit state function, measures of failure and safety targets, but also the qualitative features characterizing the building heritage are identified.

Contents

2.1 Reliability: a definition

Regarding reliability, there are several definitions which are available in literature. The one that perhaps is the most extensive and correct is given by [54], which states that reliability is the ability of a structure to comply with given requirements under specified conditions during the intended life for which it was designed; in quantitative sense, reliability may be defined as the complement to 1 of the probability of failure.
In the definition of reliability, four relevant aspects are considered, which need to be clarified for its univocal interpretation:

- the evaluation of the failure probability;
- the definition of structural failure;
- the definition of the required service life;
- the definition of the conditions of use.

In the light of the above, the definition of failure is very broad, because it corresponds to all situations where the structural performance during the intended service life is below the requirements, so that it can represent, for example, the structural collapse, in case of ultimate limit states (ULSs), or an unsatisfactory behaviour, in case of serviceability limit states (SLSs) and so on.
It must be stressed that the definition of the service life is largely notional, since it should be considered as the time interval during which the structure, provided it is subject to the routine maintenance, is able to fulfil the requirements, so that it has no "biological" meaning.
In assessing the reliability or the probability of failure it should be considered that:

- Uncertainties affecting structural performance can never be entirely eliminated; consequently, any structure may fail, although with a negligible probability, even when it has been declared reliable. It follows that reliability conceived in a black and white way, with an absolute meaning, does not exist, since in the design as well as in the rehabilitation process a certain probability that a failure may occur within the intended life of the structure has to be accepted;

- Uncertainties are modelled according to the theory of probability. Although several methods for modelling uncertainty exist in literature, they have little practical significance, as sound experimental data are available only in few cases. Therefore the above definition favours an approach using random variables and stochastic process, for which results of theoretical and experimental researches already exist.

Uncertainty is usually classified as aleatoric and epistemic, referring with the first term to an inherent randomness, and with the second one to a lack of knowledge. The probability of failure depends only on aleatoric uncertainty; however epistemic uncertainty is unavoidable, and if not limited it could lead to an erroneous estimation of the structural reliability. For this reason, it should be indirectly considered, e.g. using corrective coefficients like in the partial factor method.

2.1.1 Notional probability of failure

The failure probability that is usually computed is a notional estimate [131, 114]. In effect, it is calculated under the following conditions:

- reference set of modelling assumptions;
- standard method of calculation;
- uncertainties due to human involvement disregarded.

After all, taking into account uncertainty due to human error and human intervention is not trivial at all since they cannot be easily described by probabilistic tools, even because, for their inherent nature, they escape from statistical treatments. Because it is a notional probability of failure, it has not an absolute meaning, and it can be mainly used for code calibration purpose or in a comparative sense, when the components being compared are essentially similar (and therefore human errors have comparable influences) or the component is subjected to different load condition.

2.2 Peculiarities of the reliability assessment of existing structures

An existing structure may be assessed for several reasons, such as [90]:

- change of tenancy or use, including increased load requirements;
- concern about design or construction errors;
- concern about the quality of building materials or workmanship;
- evaluation of effects of deterioration;
- assessment of damage following an extreme loading event (like wind storm or earthquakes);
- concern about serviceability.

Although the reliability assessment of existing structures foresees the same steps of the reliability based design of new buildings, crucial differences distinguish the two procedures:

- the uncertainty considered in the design process has been realized in the built structure and they are no longer uncertainty anymore;
- target probabilities implemented in design code rules cannot be directly translated to the verification of existing structure because the link to societally acceptable risk criteria is likely to be different.

Considering that the actual structure is one realization of many possible outcomes, the uncertainty associated with the process has disappeared. However, what now replaces this uncertainty is our limited knowledge of the actual realization [114]. Theoretically speaking, given sufficient resources and ideal measurement and monitoring techniques, this lack of knowledge can be overcome. In practice, this is possible only to a limited extent. The acquirable knowledge is also different according to the characteristics of the building that has to be assessed; the main factor that probably determines what can/can not be known is represented by the age of the structure. But, on the opposite, it must be stressed that the present situation of an existing structure can be the result of complex transformations due to human or environmental interventions, which often are not easily detectable.

A classification of existing buildings is proposed in the Paragraph 2.2.1, and features characterising each category are reviewed also considering the development of the engineering practice across the centuries.

2.2.1 Classification of existing structures

For many centuries, structural and architectural features were both embraced by a single holistic design process; an adequate structural design was assured by suitable architectural canons and orders, together with regulator alignments and proportional rules based on the empirical evidence of successfully performing structures. Building techniques were generally codified on empirical bases, even if adapted to local traditions, while the materials were often of local origin. Most of the structures were made of brick or stone masonry, sometimes combined with timber elements, when greater tensile or bending capacity was required [106]. Clearly, as a structural dimensioning adopting empirical formula can be sensibly different from that we could determine today, historical structures, even not affected by structural failures, largely exhibit lack of reliability if inspected in the light of modern codes.
A time of changes was represented by the emergence of modern science during the early modern period, when developments in applied mathematics, physics and chemistry allowed to give more and more sound theoretical bases to the design and constructing process, separating the structural aspects of the building from the architectural canons and its visual appearance. Firstly, Galileo rejected the use of both the architectural types and the proportional design rules, revealing that geometric similitude does not imply mechanical similitude [26], subsequently Hooke laid the basis of the theory of elasticity and finally Navier and De Saint-Venant developed it, making possible the application of mathematical sciences in building engineering. At the same time, the outcomes of Industrial Revolution made available other advanced building materials, so that the choice, originally limited to timber and masonry, widened considerably, while easier transports and world-wide trade allowed the phenomenon to become global. It began with the large-scale production first of iron and cast iron, and then of steel; next came Portland cement, followed by a highly versatile composite material, reinforced concrete. A wider choice of construction material led to a sudden development of new structural types, often resulting ineffective or unsuccessful. In effect, from one side the methods for calculating even simple indeterminate structures were still in

an early stage; from the other side poor detailing and poor workmanship were the results of the unavoidable lack of knowledge and experience in new materials. Progress in building design and construction was therefore slow and often attributed to analogical methods and trial-and-error processes. The $20th$ century represented a new dawn for structural engineering, with the development of Codes and Standards aimed at giving a common basis for design, so assuring the achievement of target reliability level. Early Codes adopted design procedures based on the precepts of allowable stress design: the concept of an arbitrary Factor of Safety was suggested by Freudenthal [70], and the probability-based design began slowly to emerge. The first attempt to draw Limit State codes, introducing separate partial safety factors on loads and materials, dates around 1930-1935, but it was during the '70s that limit state concepts became mature. The Limit State design was not just a change in calculation format: the intention was that variations in loads, materials and member strengths would be statistically analysed and probability theory used to calculate more rational design values. The underlying aim was to produce structures with approximately uniform reliability with respect to certain limit states. However, because of the lack of relevant statistical data and the need to preserve accustomed tradition, partial factors were still largely based on past practice and subjective judgments. Nowadays the principle according to a structure should be designed to have appropriate degrees of reliability with respect to ULSs and SLSs represents a basis of the modern Codes and of Eurocodes in particular, that, together with durability and robustness requirements, gives guidance to design methods based on reliability analysis concepts, permitting also their application to special design.
According to the brief evolution of construction and design outlined above, it is possible to propose a first classification of existing structures in terms of design approach, materials and structural system:

Ancient structures (until late $18th$ century): mainly made of masonry and wood (although in Roman age some relevant examples of concrete structures can be found), designed according to architectural canons and empirical formulae;

Modern structures ($19th$ century): built with iron, cast iron, steel and reinforced concrete, often designed according to the theory of elasticity and trial-and-error processes, characterised by structural systems generally composed of trusses and frames;

Contemporary structures (from the $20th$ century until today): mainly built

(a) Contemporary

(b) Modern

(c) Ancient

Figure 2.1: Examples of different types of existing structures [170].

with steel and reinforced concrete, designed according to Codes and Standards that implement a set of partial factors, taking into account all the uncertainty involved in the design and construction process.

2.3 Basic steps of the probabilistic reliability assessment

2.3.1 Definition of failure modes and limit states

A limit state is a distinct condition separating states of a structure for which certain requirements are fulfiled or not. Usually, states are called ”safe” or ”unsafe” according to if they fulfil the requirements or not. As it has already been mentioned in the previous Chapters, two fundamental types of limit states are recognized: ULSs and SLSs. ULSs are associated with collapse and other forms of structural failure, while SLSs correspond to conditions of normal use (deflections, vibration, cracks). A component, or a system, may fail a limit state in any of a number of failure modes. Modes of failure include mechanisms such as: yielding, bursting, ovality, bending, buckling, creep, ratcheting, de-lamination, denting, fatigue, fracture, corrosion, erosion, environmental cracking, excessive displacement, excessive vibration. Once that failure modes have been identified, the limit state criterion must be expressed in mathematical terms through a performance function, that is a suitable transformation of the basic variables [41]. In the space of basic variables, "safe" and "unsafe" states are separated by the failure surface. The basic variables that should be considered are dimensions of the structural elements, load characteristics, material properties, model uncertainties. The fundamental case of the theory of structural reliability is the analysis of the simple requirements that the action effect E is smaller than resistance R.

$$E < R. \tag{2.1}$$

This condition leads to the fundamental forms of the limit state function[114]:

$$G = R - E = 0 \tag{2.2}$$

or

$$Z = \frac{R}{E} = 1, \tag{2.3}$$

assumed that $E \neq 0$, being G the so-called *safety margin* and Z the so called *safety factor* (Fig. 2.2).

2.3.2 Measures of failure

2.3.2.1 The failure probability

Both E and R are generally random variables and it is necessary to accept the fact that the limit state may be exceeded with a certain probability. The measure of failure is then the probability associated with the event $R - E \leq 0$ or equivalently $R/E < 1$. Reliability is then defined as the complement of the probability of failure. Considering that R and E are characterised by a joint pdf $f_{R,E}(r, e)$, the probability of failure p_f associated with the margin G is:

$$p_f = \int_{(r-e\leq 0)} f_{R,E}(r, e) dr de. \tag{2.4}$$

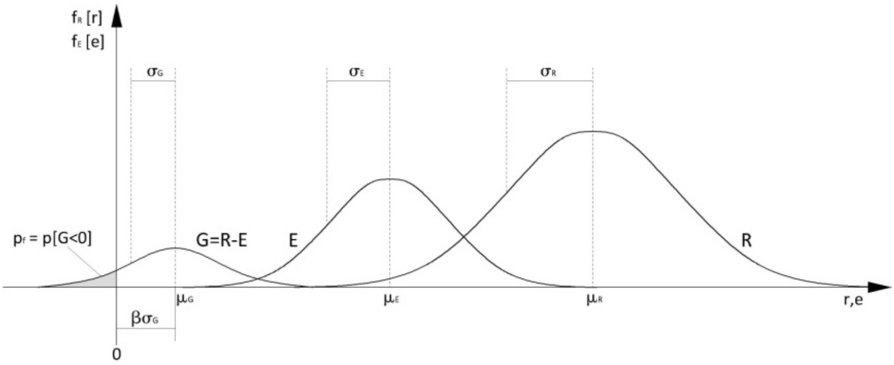

Figure 2.2: Basic R-E problem: qualitative representation of $f_R(r)$, $f_E(e)$ and $f_G(g)$.

The integral can be calculated according to direct integration, but it can be analytically computed only in simple cases; more generally, it is solved by numerical integration or Monte Carlo (MC) simulation.

2.3.2.2 The reliability index

Since the probability of failure is generally a small number, a reliability index that can be more easily handled is introduced. The first reliability index has been introduced by Cornell [14] and defined in the following way:

$$\beta_C = \frac{\mu_G}{\sigma_G} = \frac{1}{V_G}. \tag{2.5}$$

β_C basically indicates the distance, in terms of σ_G units, between the mean point μ_G and the limit state. In case of $G = R - E$ it holds:

$$\beta_C = \frac{\mu_R - \mu_E}{\sqrt{(\sigma_R)^2 - 2\rho_{RE}\sigma_R\sigma_E + (\sigma_E)^2}}, \tag{2.6}$$

where μ_R, μ_E and μ_G are the mean values of R, E and G, respectively, σ_R, σ_E and σ_G are the standard deviations of R, E and G, respectively, and ρ_{RE} is the correlation coefficient of R and E, which is nil for uncorrelated variables. If R and E are also Gaussian, then it follows:

$$p_f = \Phi(-\beta_C), \tag{2.7}$$

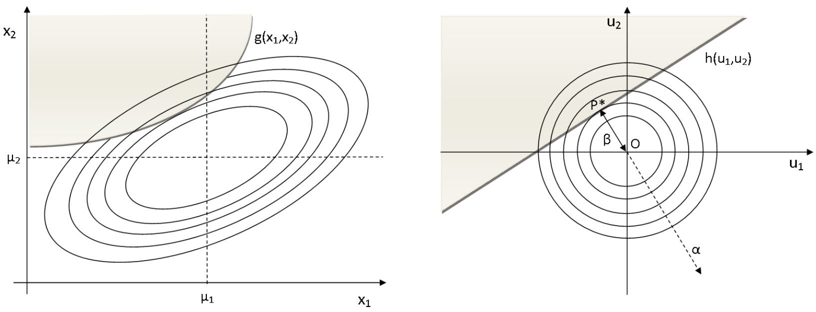

Figure 2.3: The physical space of the random variables x_1 and x_2 (on the left) and a representation of the Hasofer-Lind index in the standard space (on the right).

where Φ is the standard normal integral. The equation basically represents a change of scale of p_f into a number that can be better appreciated. However, β_C has the following drawbacks:

1. Non-invariance of β_C in various representations of the same limit state;
2. Eq. 2.7 depends on the distribution of G when R and E are not Gaussian or/and the limit state is non-linear.

In order to solve these problems, Hasofer-Lind [78] proposed to transform the variables into a new space of statistical independent Gaussian variables through an isoprobabilistic transformation. Then the reliability index β_{HL} is defined as the distance in the standardised space U between the origin O and the point P^* of the limit state surface closest to the origin. P^* is the point in the limit state surface corresponding to the maximum density of probability in the standardised space U. It is the design point, considered from the point of view of partial coefficients. β_C can be considered as an approximation of the exact and invariant β_C by a first or second order development around the point P^*. The determination of P^* represents a non-linear optimization problem, that can be solved according to several algorithms [47]. The calculation of p_f then depends on the level of approximation and the characterization of the limit state; two methods are especially described in the following.

FORM The First Order Reliability Method (FORM) approximation consists of associating to the value of β the probability $\Phi(-\beta)$; the real limit state in the standardised space is basically replaced by a hyperplane at the design point P^*. Once that P^* has been found, the hyperplane is perpendicular to the vector P^*O of direction α. The accuracy of the FORM approximation depends on the form of the limit-state at the most probable failure point. The exact solution is obtained only in case of linear limit state and Gaussian random variables. If the concavity of the limit state is toward the origin, then p_f is higher than that one obtained with the FORM approximation; if instead it is convex, p_f is lower.

SORM The defect of the FORM is given by the curvature of the limit state at the point P^* in the standardised space. Methods that deal with the nonlinearity of the limit state function has been termed 2^{nd} order. Usually, a parabolic, quadratic or higher order surface is fitted to the actual surface, centred on the design point. Some decision about the extent to which the approximation is valid away from the design point should be taken. In order to estimate the probability content enclosed by a quadratic surface, two methods have been proposed, one based on sampling, the other one based on asymptotic concepts. In the first case, it is possible to sample in the space between the linear and the quadratic approximation or to use the result of FORM approximation as a starting point for simulation about the error that one makes by considering the linear limit state. In the second case, p_f can be estimated from the determination of the limit state curvature at the point P^* and applying an asymptotic formula. However, the limit state must be slightly non-linear, continuous and twice differentiable, as well as it should not contain a lot of random variables. An alternative approach is to fit a quadratic surface through not only P^* but also a set of points on the limit state surface with arbitrary location. This approach is likely the response surface method that will be discussed in Chapter 4.

2.3.3 Safety targets

The definition of acceptance criteria in the reliability assessment of existing structures is one of the most crucial but still controversial issues. Acceptance criteria are given in terms of probability of failure or reliability index. Verification thus consists in the comparison of the actual reliability and the target safety level. A structure is declared reliable if the following condition is verified:

$$(1 - p_f) > (1 - p_d) \tag{2.8}$$

or, equivalently,

$$\beta > \beta_t, \tag{2.9}$$

where the target reliability is represented by $(1 - p_d)$ or β_t. As it has already been said in Chapter 2.2, target probabilities for the assessment of existing structures may be different from those of new one, but only when the existing structure behaves satisfactorily and no intervention is planned on it. However, since the formers are often defined considering the latters as a starting point, a brief review of the safety targets adopted in design will be given. Safety targets for new structures can be usually found in Codes and Standards, or defined through analytic formulas. In the first case, they have been derived by directing studying structural members. The so obtained values of β_t depend on many factors (such as the type of components, loading conditions, structural materials), and consequently have a great scatter; but the result of any reliability study also strongly depends on the probability models used to describe the basic variables. Since models are not unified and used systematically, the proposed targets are thus average values of the reliability level characterising existing structures.
Tables that can be found in [54, 89, 90] are reported in the following (Table 2.1, 2.2, 2.3). In [54] reference targets are given for different reliability classes, which depend on the economic, social, environmental and human life consequences of a structural failure. In [89, 90] reference targets are given for different magnitude of the failure consequences and relative costs of the safety measure.

It appears that all those documents do not provide a direct and explicit guidance on how to take into account T_d. Although [54] defines a couple of reliability indexesfor two different reference periods T_d (1 year and 50 years) and values given by [88] refer to $T_d = 50$ years, no explicit rules are offered for the adjustment of β to different T_d. Some indications are given in [90], where it is suggested to multiply the annual p_f related to T_d by a factor c accounting for the dependence of different failure events within 1 year, and in the [89], where it is stated that reference values for different T_d should not sensibly changed. According to [44], on the basis of extreme value theory the reliability index for a period of n years maybe calculated from the following

Reliability class	Consequences for loss of human life; economic, social and environmental consequences	β_a for $T_a = 1$ year	β_d for $T_d =$ 50 year	Examples of buildings and civil engineering works
RC3-High	High	5.2	4.3	Bridges, Public buildings
RC2-Normal	Medium	4.7	3.8	Residential and Office buildings
RC1-Low	Low	4.2	3.3	Agricoltural bildings and greenhouse

Table 2.1: Reliability classification [54].

	Consequences			
Relative cost of safety measure	Small	Some	Moderate	Great
High	0	1.5	2.3	3.1
Moderate	1.3	2.3	3.1	3.8
Low	2.3	3.1	3.8	4.3

Table 2.2: Target reliability index β_t for the design working life T_d [89].

	Consequences		
Relative cost of safety measure	Minor	Moderate	Large
Large	$\beta = 3.1$	$\beta = 3.3$	$\beta = 3.7$
Normal	$\beta = 3.7$	$\beta = 4.2$	$\beta = 4.4$
Small	$\beta = 4.2$	$\beta = 4.4$	$\beta = 4.7$

Table 2.3: Target annual reliability for ULSs [90].

approximate equation:

$$\Phi(\beta_{t,n}) = (\Phi(\beta_{t,1}))^n, \tag{2.10}$$

that in case of small probabilities can also be written as:

$$\Phi(\beta_{t,n}) = \frac{\Phi(\beta_{t,1})}{n}. \tag{2.11}$$

It is another story with the definition of the safety targets through mathematical expression, since T_d represents a factor entering into the equation. According to [30], target probabilities can be defined in the following way:

$$p^*_{fN} = 10^{-4}\mu t_L n^{-1}, \tag{2.12}$$

where t_L is the structural design life in years, n is the average number of people within or near the structure during the period of use and μ is the social criterion factor (see [114]. A different proposal is made by [4]:

$$p^*_{fN} = 10^{-5} A W^{-1} t_L n^{-\frac{1}{2}}, \tag{2.13}$$

where A and W are the so-called activity and warning factors respectively. The values of all those coefficient can be taken by [114], but they should be considered as roughly indicative. Comparison of the two approaches is not possible; furthermore, consideration of injuries, economic costs of failure and the number of people that could be involved in the structural failure are not only a very complicated matter, but also outside the possibility of the intervention of the designer, which is obliged in any case to grant the reliability level required by the Code. As it has already been noticed, several difficulties could arise in defining p_d and β_t without any reference to the context in which reliability is estimated. For this reason, values obtained directly from backcalculation varying in the range of $3 - 3.5$ and referred to $T_d = 50$ years can be considered as reference for any studies, provided that coherent hypotheses are assumed.

2.3.3.1 Safety targets for existing structures

In case of assessment of existing structures, defining reliability targets is more difficult and controversial. In effect as the assessment process differs significantly from the design of new structures, such as the cost of safety measures and the cultural and historical value of the structure [43]. Because of

Consequence class	Minimum reference period	$\beta_{t,new}$		$\beta_{t,ex,u}$		$\beta_{t,ex,r}$	
		wn	wd	wn	wd	wn	wd
0	1 year	3.3	2.3	2.8	1.8	1.8	0.8
1	15 year	3.3	2.3	2.8	1.8	1.8	0.8
2	15 year	3.8	2.8	3.3	2.5	2.5	2.5
3	15 year	4.3	3.3	3.8	3.3	3.3	3.3

Table 2.4: Required β – values for the minimum reference period [153, 152]. Class 0 is like class 1 but no human safety is involved. wn stands for 'wind non dominant', while wd is 'wind dominant' case.

economic constraints, in several studies a reduction of the safety level usually considered for new structure is claimed. The safety target $\beta_{t,ex,u}$ under which the structure is unfit to use can be thus defined in the following way [153]:

$$\beta_{t,ex,u} = \beta_{t,new} - \Delta\beta, \tag{2.14}$$

where $\beta_{t,new}$ is the target for new structure as given by Codes and Standards and $\Delta\beta > 1$ (usually 1.5). However the limit value introduced by human safety becomes decisive in most of the cases. For this reason a minimum value of $\beta_{t,ex,u} = 2.5$ is required for all the situation, in combination with a minimum reference period of 15 years, at which corresponds $p_f = 10^{-4}$. The $\beta_{t,ex,r}$ for which the structure does not need any repair intervention can be defined according to Eq. 2.14 but considering $\Delta\beta = 0.5$, that is approximately the difference between the safety level of previous Codes and new Codes. Basically with a reduction of 0.5 we avoid that structures designed according to old building Codes and at that time considered safe enough have to be replaced or radically changed. Those concepts are summarised in Table 2.4 presented in [153], where lower safety targets are also considered in case of wind dominant. This suggestionis due to the fact that backcalculation of existing structures considered safe enough under wind load reveals lower reliability. However other factors might influence that result, such as the implemented transfer functions that lead to the action effects and the applied model for the wind force. In author's opinion further researches should be carried out on this topic. Therefore, assuming $\beta_n = 3,8$ referred to $T_d = 50$ years, $\beta_t = 3,3$ is thus obtained, that again is in the range of 3-3,5 suggested by backcalculation. In case of historical buildings whose cultural and historical value needs to be preserved, [142] proposed the following formula, suitable for accounting of the social importance of the building, for defining target probabilities:

$$p^*_{fN} = \frac{10^{-4} S_C t_L A_c C_f}{n_P W}, \tag{2.15}$$

where S_c is the societal criterion, A_c is the activity factor, C_f is the cost factor, n_P is the number of lives put to danger, W warning factor in case of failure.

2.4 Summary

In this introductive chapter, the key elements involved in the reliability assessment of existing structures have been presented. At first, a definition of reliability is given; in order to do this, the concepts of limit states, measure of failure and safety target have been introduced. Paralleling to those technical issues, other aspects are tackled, especially the variety of existing structures and the motivation behind their assessment. This chapter intends to identify the main factors that should be considered approaching the reliability assessment of existing structures.

Chapter 3

Construction of stochastic models

An important step in the reliability assessment of existing structures is the definition of the probability models for the basic random variables. Since the information usually available are both qualitative and quantitative, the Bayesian analysis is proposed. Attention is especially drawn on the choice of the prior distribution and the computational aspects of the posterior distribution.

Contents

3.1 Introduction

3.1.1 Different types of random variable

A fundamental step of the reliability assessment of existing structures is the definition of the probability models for the random variables of the limit state function. Let us remark that any stochastic phenomenon is by nature complex and one may decide to consider and thus model the complexity or not. In both cases, the variability is described by a function, but in the second case the function belongs to a certain family, and it can be defined by setting some parameters. In the present work, only parametric distributions are considered. Typical examples of basic variables are dimension, density or unit weight, material load, material strength [114]. It is important to underline that, generally speaking, variables can be of different types [100]:

- elementary deterministic or random variables. These are the lowest level variables, declared by a simple assignment of a numerical value or by associating a *pdf*;
- random variables depending on deterministic variables. These variables are declared by a function involving other variables which remain deterministic in the entire tree up to the level of deterministic variables;
- random variables that are a function of at least one random variable. These variables depend on random variables in the argument tree;
- variables that are the results of the finite element calculation, such as the stresses at given points, a displacement etc. etc.;
- variables whose distribution parameters depend on other random variables. These variables are also called 'compound variables', whose distribution parameters are defined by explicit functions of other random variables in the model.

In the elementary case, the limit state is defined by two random variables, resistance and action effect, characterised by their own *pdf*. However in the majority of situations, several random variables are involved, and it results that resistance is a variable of at least one random variable while action effect is a result of FEM calculation.

3.1.2 Different types of uncertainty

As [114] has reported, a range of uncertainty affects the reliability assessment problem. It is not possible to express through *pdf*s every type of uncertainty, therefore attention will be mainly focused on those that can be modelled in such a way: model, physical, and statistical uncertainty. Model uncertainty concerns the vagueness in the representation of physical behaviours, such as through the limit state equation, due to a lack of knowledge. It can be incorporated into the reliability analysis by introducing a variable to represent the ratio between the actual and the predicted model response or output. Physical uncertainty is generated by the inherent randomness nature of the variable; although sometimes it can be reduced (for example increasing the number of data or increasing the effort in quality control, like in the case of material strengths), it cannot be completely eliminated, as it is evident for natural phenomena. *Pdf*s represent the proper tool to handle this type of uncertainty, although data might be scarce and the variable must be assessed subjectively. Statistical uncertainty affects statistical estimators and it is produced by limited sample or bias in the data recorded. It can be incorporated in the reliability analysis by letting the parameters themselves be random variables. Similarly to modelling uncertainty, it is also due to a lack of knowledge and it can be reduced by increasing the sample size or reducing the bias.

3.1.2.1 Aleatoric vs. epistemic uncertainty, frequency vs. subjective probability

As it has been already sporadically mentioned in Chapter 2, uncertainty can be grouped into two main class, aleatoric and epistemic [38]. The word aleatoric, that comes from 'alea', meaning in latin a die, is used to identify the uncertainty due to randomness; the word epistemic, that is instead greek and means 'pertaining to knowledge', is due to imperfect knowledge about something, that is not in itself random and is, in principle, knowable [133]. The difference between aleatoric and epistemic uncertainty is often paralleled by the distinction between frequency and subjective probability, the last one also called Bayesian.
It is necessary now to say a few words regarding the concept of probability. In a famous quote attributed to Bertrand Russell, the mathematician goes directly to the point:

> Probability is amongst the most important science, not least because no one understands it.

Although probability might be impossible to understand, Richard Feynman raises the bar a bit higher, when in his famous lesson about probability reminds us that we speak of probability only for observations that we contemplate being made in the future [64]. Probability is thus our best estimate of what would occur in a certain number of imagined observations; it depends, therefore, on our knowledge and our ability to make estimates, and for this reason, it is always subjective and based on our common sense.
However, we can speak of probability for observations having different characters. In particular, observations can be more or less repeatable, or in other words, they can appear to be more or less equivalent for our intended purposes. In engineering, we usually use the word 'frequentistic' to call those probabilities related to observations that can be easily repeated to our intended purposes, and originated by a process having intrinsic randomness; we indeed use the word 'subjective' or 'Bayesian' to call those probabilities related to one-off unrepeatable observations.
While the first definition is usually applicable to aleatoric uncertainty, the second one is used for epistemic uncertainty, although in some cases epistemic uncertainty can be somehow evaluated in a frequentistic manner – think for example to bootstrap and jackknife techniques. It is indeed possible to resort to subjective probability in order to express the degree of belief related to aleatoric uncertainty.
In several engineering problems, and especially in the reliability assessment of existing structures, both types of uncertainty coexist: however only the aleatoric uncertainty takes part in the determination of the probability of failure. If we wish to perform sound reliability assessment, it is important to find a framework in which both uncertainties can be easily handled and combined, and epistemic uncertainty, that might corrupt the final results, can be reduced when new data become available. The answer is represented by the Bayesian analysis.

3.1.3 Population, sample and statistical inference

When aleatoric uncertainty is considered, the concept of population becomes fundamental, referring with this term to the entire group of elements the investigator wants information about. The investigator is interested in the distribution of the population, especially in the population parameters. Often it is not feasible to get information about all the units in the population because the population may be too big. For this reason, the investigator draws a sample from the population and gets information from the individuals in that sample. Sample statistics are calculated from sample data. They are numerical characteristics that summarize the distribution of the sample, such as the sample mean, median, and standard deviation. A statistic has a similar relationship to a sample that a parameter has to a population. However, the sample is known, so the statistic can be calculated. Statistical inference is thus the act of making a statement about population parameters on the basis of sample statistics. In order to infer properly the population parameters, the sample must be representative of the population as a whole, and thus the distribution of the sample must be similar to the distribution of the population from which it came. Sampling bias, a systematic tendency to collect a sample which is not representative of the population, must be avoided. It would cause the distribution of the sample to be dissimilar to that of the population, and thus lead to very poor inferences.

3.2 Bayesian inference

3.2.1 Bayes' theorem

Let us consider $A_1, A_2, ..., A_n$ disjoint events whose union has probability one, at which prior probabilities $P(A_i)$ are associated. Suppose that an event B occurs, for which $P(B|A_i)$ (the likelihood of the event B given A_i) is known for each A_i. The Bayes theorem states that:

$$p(A_i \mid B) = \frac{p(B \mid A_i)p(A_i)}{\sum_{j=1}^{n} p(B|A_j)P(A_j)}. \tag{3.1}$$

The theorem, firstly proposed by Thomas Bayes in his famous paper [10], represents an actualization principle: it describes the updating of $p(A_i)$ to $p(A_i|B)$ once that the event B has been observed. Thomas Bayes actually proved a continuous version of this result, namely, that given two random variables X and Y, with conditional distribution of X given Y $f(x|y)$ and marginal distribution $g(y)$, the conditional distribution of Y given X is:

$$g(y \mid x) = \frac{f(x \mid y)g(y)}{\int f(x \mid y)g(y)dy}. \tag{3.2}$$

Bayes' theorem expresses conditional probability or conditional density functions. However, depending on the interpretation of probability, the meaning of Bayes' theorem differs significantly. If probability (or density) is interpreted in a frequentistic manner, the theorem expresses the proportion of an event given the occurrence of another event. Conversely, if probability reflects the relative plausibility or degree of belief attributed to a certain event, Bayes' theorem forms the mathematical basis for adjusting or updating the probability as more evidence becomes available.

3.2.2 Bayesian statistics

Laplace [99] went further and considered that the uncertainty on the parameters Θ of a model could be modelled through a probability distribution π on Θ, called prior distribution. The inference is then based on the distribution of Θ conditional on x, $\pi(\theta \mid x)$, called posterior distribution and defined by:

$$\pi(\theta \mid x) = \frac{f(x \mid \theta)\pi(\theta)}{\int f(x \mid \theta)\pi(\theta)d\theta}. \tag{3.3}$$

In a Bayesian statistical model, statistical parameters are also random variables characterised by a *pdf*, that has to be interpreted in a subjective way [42]. Therefore, while in the frequentistic approach, statistical parameters are represented by fixed quantities that are inferred from the data through classical statistical techniques such as Maximum Likelihood Estimation or the Method of Moment, in the so-called Bayesian approach parameters are represented by random variables whose *pdf*s should be interpreted as a degree of belief. This fact is not only revolutionary but also it resonates in a number of different modes. First of all, with Bayesian statics we basically defined a two-level hierarchical model, in which the second level is represented by the *pdf* on the statistical parameters of the probability model at the first level: the statistical parameters of the second level distribution are usually called 'hyperparameters', and the *pdf* 'hyperdistribution'. Secondly, in doing so we move a step forward into the field of imprecise probability, that attempts to model the vagueness affecting probability models. Thirdly, the degree of belief can be revised using the rules of probability when new data becomes available, and once that the posterior distribution has been defined, statements about the parameter can be made.

3.3 Defining prior distributions

3.3.1 Preamble

The definition of prior distributions depends on the framework in which the Bayesian analysis is applied, and the scopes that are pursued. In this work two scopes are especially considered:

1. the improvement of probability models for the random variables involved in the reliability assessment;

2. the improvement of the input parameters of a FEM analysis.

In effect, the improvement of the input parameters of a FEM analysis results in better probability models for the outputs, that usually are action effects taking part to the limit state function. Therefore the second point leads in practice to the first. Without entering into the details of the reliability assessment associated with FEM analysis, it is worth to mention here the fundamental difference between reliability and FEM analysis:

- Reliability analysis is by nature a probabilistic analysis, in which parameters are represented by random variables characterised by a *pdf* and whose result is expressed in probabilistic terms; the *pdf* should describe the aleatoric quota of the related uncertainty, but usually also epistemic uncertainty is present, affecting the probability model that describes the inherent randomness.

- FEM analysis is by nature a deterministic procedure for determining the response of the structure under specified load condition, as it has been also observed by [97]. This means that input parameters are characterised by a unique value, that in case of load and material characteristics can be considered representative of the whole structure or parts of it [63]. The model is then considered as a black box: if a given set of input parameters is selected, running the model provides a unique response vector; furthermore, running twice the model using the same input vector will yield exactly to the same output [157].

While in the first case the prior distribution describing the basic variables incorporates both types of uncertainty, the epistemic and the aleatoric quota, in the second case it represents exclusively the epistemic uncertainty. Furthermore, while in the first case it is opportune to resort to Bayesian statistical model (and in doing so to separate the aleatoric part to the epistemic part), in the second case uncertainty can be represented by an unique distribution, that expresses the degree of belief that a certain value can be representative of the FEM.

3.3.2 Who is the expert?

Elicitation is the action of quantifying the knowledge and the judgment of any person, that in this specific context acquires the status of 'expert'. Expertise involves not only having a great knowledge of a subject matter but also how the person organises and uses that knowledge. The elicitation process comprises several steps, that can be carried out by different persons:

- the person that needs the result of the elicitation;
- the expert that has knowledge regarding uncertain quantities of interest;
- the statistician, who because of its probabilistic training validates the results and provides feedback.

In practice one person may play more than one role; in the current work, it is assumed that the engineer is the person that holds or collects prior knowledge and organises it in such a way that prior distribution can be defined and thus used to perform the reliability assessment. It is assumed that he has not only a deep understanding of reliability methods but also some training in statistics as well as some knowledge regarding existing buildings acquired over the years.

3.3.2.1 Acquirable knowledge and its representation

Let us first go through the information that is desirable to acquire approaching the assessment of existing buildings. This knowledge can be at the beginning classified into two main groups: site-specific and non-site-specific information. The first group comprises outcomes of an investigation carried out because of the assessment, that might be qualitative or quantitative by nature. Examples might be results of static or dynamic tests carried out on the structure, material mechanical tests on samples collected in situ but executed in a laboratory, monitoring, geometrical surveys, endoscopy and thermography. This group also includes the results of studies carried out during the erection of the structure or immediately after, such as those obtained from material acceptance control or acceptance tests on the structure. Actions suffered by the structure over the years such as earthquakes, floodings and fires, also represents relevant knowledge related to the site where the structure stands. In case of ancient structures, site-specific information can be obtained carrying out archive studies, where project drawings, contracts and specifications concerning the construction of the building, contracts and bills referring to production, sale and purchase of building materials, judgements or court deeds regarding controversies in the building sector, and so forth might have been collected. In case of contemporary structures and sometimes modern building, another important source of site-specific information is represented by original drawings and calculation collected in public archives or preserved by the architect or engineer who designed the structure - and survived until nowadays.

Non-site-specific information can, in turn, be separated into two main sets, that are those originated from Codes and Standards implemented when the structure was erected, as well as manuals and essays gathering practical construction rules, and those developed during scientific research, that can be found in scientific papers, reports and books, or simply collected in modern database of records or online archives. Scientific researchers comprise studies regarding specific structures or groups of similar buildings, field study aimed at collecting data about environmental actions, laboratory tests on material or structural elements, computer simulation aimed at investigating structural behaviour. Researchers can be carried out in an academic environment but also by private companies. The object of scientific researchers might be not only contemporary structures but also modern and ancient buildings: in the latter case the so-obtained results often represent the only source of knowledge since poor information is generally available regarding any building built before the $18th$ century in absence of any formal design; more information is indeed available regarding modern structures, since it is possible to refer to original guidelines and original companies, as well as engineering textbooks and publications on professional journal and newspapers. Clearly, Codes and

Standards are developed not only considering established construction rules and practice, but also the outcomes of scientific investigations. However, a substantial difference exists between Codes and scientific essays, in the way in which knowledge is organised. While results of scientific investigation are usually displayed in the form of crude data or statistics, Codes and Standards summarizes knowledge in terms of representative values and safety coefficients. Regarding material mechanical parameters and actions, the most popular representative value is the characteristic value, that corresponds to the 5 % superior or inferior distribution fractile, depending if it refers respectively to action or resistance (and in case of action it often corresponds to a value having a certain return period).

3.3.3 Eliciting prior distributions

3.3.3.1 Understanding the parametric family

The first step toward the definition of a *pdf* is to identify the parametric family to which it belongs. It is worth to mention the so-called Physicist's approach, according to which the variability of the parameter can be established based on physical reasoning and asymptotic concepts that allow expressing the element properties through few essential characteristics. An example is represented by the case in which the random variable consists of the sum of many other variables: in this circumstance the Central Limit theorem can be invoked, according to which the *pdf* of the sum of a large number of random variables approaches the normal distribution, irrespective of the individual distribution of the random variables [114]. This reasoning might be appropriate for material compressive strength and for dead load. In case of material, the behaviour on a macroscopic scale is based on the state of this material at a meso- or microscopic scale [100]. The object is thus taken as an arrangement of a large number of small elements whose statistical properties as well as their type of interaction can be assessed qualitatively; otherwise, it is possible to define a reference volume, that generally corresponds to some specified test specimen volume at which material testing is carried out. This volume generally neither corresponds to the volume of the virtual strength elements introduced at the micro-scale modelling level nor to a characteristic volume for in situ strength. Once that a reference volume has been identified, guidance on the type of distribution may be obtained by assessing its performance in terms of some microsystem behaviour. The performance of test specimens, regarded as a system of microelements, can usually be interpreted by one of the following strength models:

- Weakest link model;
- Full plasticity model;
- Daniel's bundle of threads model.

When applying these models to systems with increasing number of elements, they generally lead to specific distributions for the properties of the system at the mesoscale level. The weakest link model leads to a Weibull distribution, the other two models to a normal distribution. For larger coefficients of variation, the normal distribution must be replaced by a lognormal distribution in order to avoid physically impossible, negative strength values [90]. In case of action such as wind, the maximum wind velocity per year might be represented by the Gumbel distribution, the limiting distribution for the largest random variable distributed according to the Pareto or Cauchy distribution. In effect, the wind speed is based on an underlying wind phenomenon which is described essentially by a normal *pdf*; since the normal *pdf* is a special case of the Cauchy distribution, the Gumbel distribution might be chosen.
Physical reasoning also comprises speculation about the ranges of possible values assumed by the random variable; here three principal cases arise:

1. unrestricted random variable. If the random variable can take any value, positive or negative, also the family of distributions that will be chosen to represent uncertainty must have the same property. Examples are the normal or Student's t-distribution;
2. positive random variable. Similarly, if the random variable must be positive, or non-negative, then suitable distributions having this property are the gamma, inverse-gamma, lognormal and F-families. This situation arises for example when the random variable is a population variance;
3. bounded random variable. The most widely used distributions are those belonging to the Beta family.

3.3.3.2 Subjective determination of the prior distribution

The problem of constructing wholly subjectively a prior density is considerably difficult, therefore useful techniques will be discussed in the following. Assuming that a given functional form has been selected, it is necessary to choose the values of the parameters that characterize the functional form best matching the prior belief. The easiest way of subjectively determining prior parameters is to calculate them from estimated prior moments, as it is the case of the beta distribution. Unfortunately, the estimation of prior moments is often an extremely uncertain undertaking. The difficulty lies in the fact that the tails of a *pdf* have a drastic effect on its moments, for example when dealing with unbounded (or very large) parameter spaces. For bounded parameter spaces, the subjective determination of prior moments is more reasonable, since the tails have a minor effect on the moments. A better method for determining prior parameters is to subjectively estimate several fractiles of the prior distribution, and then choose the parameters of the given functional form to obtain a density matching these fractiles as closely as possible. Since it is precisely the determination of fractiles which is easiest to do subjectively, this approach is considerably more attractive than the moment approach. Furthermore, for a given functional form, only a small number of fractiles need typically to be found to determine univocally the prior distribution. There are available many tables of fractiles of standard densities which facilitate the application of this approach. Another method for determining the parameters of a given functional form is the so-called 'technique of equivalent sample size' or the 'device of imaginary results'. An example of the application of this technique is given in [15] considering a normal distribution. Basically, the idea of equivalent sample size is to determine a n^* such that a sample of that size would make the sample mean as convincing an estimate for the parameters as the subjective guess of the population mean. Then an appropriate prior variance would be $1/n^*$. This approach, though interesting conceptually, has two major drawbacks. First, it is helpful only when certain specific (and often unsuitable) functional forms of the prior are assumed. Second, people who are not extremely well trained in statistics have not good judgements concerning the evidence conveyed by a sample of size n, and especially they tend to underestimate the amount of information that it carries.

3.3.3.3 Noninformative and maximum entropy prior distributions

In this paragraph, two methods according which it is possible to use the Bayesian approach when no or limited prior information is available are presented. These methods are based on the definition of non-informative and maximum entropy prior distribution. A noninformative prior distribution is a *pdf* that contains no information about Θ, or in other words which favours no possible values of Θ over others. It frequently happens that the noninformative prior is an improper prior, namely one which has infinite mass. If the parameter that we would like to assess has a finite domain of n elements, the noninformative prior consists in giving to each possible value the probability $1/n$. If the domain is infinite, constructing a noninformative prior is not so trivial and one must speculate over the structure of the problem that has to be solved. In this case, it is helpful to consider transformations or reformulations of the problem which should not affect the noninformative prior.
Maximum entropy priors are indeed invoked when partial prior information is available, outside of which it is desired to use a prior that is as noninformative as possible. A helpful method of dealing with this problem is through the concept of entropy. Entropy measures the amount of uncertainty inherent in the probability distribution and it has a direct relationship to information theory. Applying this approach, the distribution which maximizes entropy among all the distributions that are plausible given the available prior information is sought. As in the case of noninformative prior distribution, finding the maximum entropy prior is easier for discrete random variables and much more difficult for continuous ones. In [15], both approaches are discussed, and proposals for the definition of noninformative prior distributions and maximum entropy prior distributions in case of continuous random variables are suggested.

3.3.3.4 Determination of the prior distribution from past data

In some cases the parameter Θ is truly a random variable and data about past determination of Θ is available. Past data can be thus used to define the prior density $\pi(\theta)$, resorting to the 'empirical Bayes' field of statistics. Two general cases can be here distinguished, that require different approaches:

- values θ_1, θ_2, θ_3, ..., θ_n of the parameter Θ from past similar situations are available. In this case, the problem is simply the standard statistical problem of determining a density from a series of observations of the density. Three possible techniques for doing so are especially available: (i) construct a histogram and sketch a reasonable prior density; (ii) assume a plausible functional form for the prior and estimate the parameters of this functional form from the θ_i; (iii) use the discrete density estimate $\hat{\pi}_0$ which gives probability $1/n$ to each of the points θ_i.

- data x_1, x_2, x_3, ..., x_n from independent random variables X_i distributed according to densities $f(x_i \mid \theta_i)$ are available, where θ_i are from a common prior density $\pi(\theta)$. In this case, the recovery of prior information about the statistical parameter might be difficult since data is distributed according to the marginal density $m(x)$. Also here we differentiate between two possible cases: (i) great variation of the data for fixed values of the parameter; (ii) small variation of the data for fixed values of the parameter. In the first case, it may suffice to use an estimate of the marginal density $m(x)$ as the estimate of $\pi(\theta)$. This is reasonable when $f(x \mid \theta)$ is thought to be quite concentrated compared to $m(x)$. The rationale for using an estimate of $m(x)$ is that the variation in the X_i is due to two sources: the variation in the θ_i (described by π), and the variation of X_i for fixed θ_i (described by $f(x_i \mid \theta_i)$). In the second case, two other approaches are available, namely the use of a given functional form and the distance method. The first approach consists in defining the parameters of the functional form estimating the moments $m(x)$ using the x. The second approach indeed consists in estimating the marginal density directly and from that the distribution of the parameters of interest; however only an estimation of the distribution parameters can be found, that leads to an estimation of the marginal density close to that one obtained with the data. A measure of the distance between those two estimations related to the concept of entropy can be established. It is thus possible to find the distribution of the parameter that minimizes that distance applying simple algorithms. For further details and exhaustive explanations about these topics please refer to [15].

Determination of the prior distribution from past data and subjective judgments It may happen that both past data and prior information arisen from another source are available for the parameter of interest. In this case, it is desirable to establish a prior *pdf* based on both subjective and past data information. However, there are no established methods for doing so. An approach consists in determining the subjective prior and the past data prior ignoring in each case the other source of information; then deciding upon a N for which the degree of confidence in the subjective prior would be equivalent to the degree of confidence in the past data prior given N observations. If n is the actual number of past observations used in constructing π_D, a natural choice for the combined prior π is then

$$\pi(\theta) = \frac{N}{N+n}\pi_S(\theta) + \frac{n}{n+N}\pi_D(\theta). \tag{3.4}$$

This approach based on a weighted average can also be applied for combining subjective belief and past data in estimating the prior parameters of a given functional form.

3.3.3.5 Hierarchical prior distributions

Hierarchical probability models have been briefly mentioned in Chapter 3.2.2; hierarchical prior distributions are thus introduced here. We have a hierarchical prior distribution when both structural and subjective prior information are available at the same time [16]. The hierarchical approach is most commonly used when the first stage Γ consists in priors of a certain functional form:

$$\Gamma = \left\{\pi_1(x|\theta) : \pi_1 \text{ is of a given functional form and } \theta \in \Lambda\right\} \tag{3.5}$$

and the second stage consists in putting a prior distribution on the hyper-parameter Θ. Such second stage prior is called 'hyperprior'. For example, in the empirical Bayes scenario, structural knowledge that x_1, x_2, x_3, ..., x_n are independent and identically distributed (i.i.d.) led to the first stage prior description:

$$\pi_1(x) = \prod_{i=1}^{n} \pi_0(x_i). \tag{3.6}$$

Please notice that any hierarchical prior can be written as a standard prior:

$$\pi(x) = \int \pi_1(x \mid \theta)\pi_2(\theta)d\theta. \tag{3.7}$$

The hierarchical structure is thus a convenient representation for a prior, rather than an entirely new entity [16].

3.3.3.6 Conjugate prior distributions

Introducing the topic of conjugate priors, we also imply another important theme that will be discussed in Chapter 3.4, that is how posterior distribution can be in practice obtained. Generally speaking, calculating posterior distribution has been a difficult task until a few decades ago, before that computers became easy-affordable tools for everyone. For this reason, approaches have been sought in order to facilitate the updating of the prior distribution. The most popular approach consists in the choice of a particular functional form for the prior distribution, given the form of the likelihood. These are the so-called conjugate priors, for which a rigorous definition is given in the following:

> Let $\mathcal{F}$ denote the class of density functions $f(x \mid \theta)$ (indexed by θ). A class $\mathcal{P}$ of prior distributions is said to be a *conjugate family* for $\mathcal{F}$ if $\pi(x \mid \theta)$ is in the class $\mathcal{P}$ for all $f \in \mathcal{F}$ and $\pi \in \mathcal{P}$.

Conjugate prior distributions have the appealing features of allowing one to begin with a certain functional form for the prior and end up with a posterior distribution of the same functional form, but with parameters updated by sample information. The advantage of this approach is that there is no need for calculating the normalizing factors, namely the evidence of the data [15]. For a given class of densities $\mathcal{F}$, a conjugate family can frequently be determined by examining the likelihood functions $L_x(\theta) = f(x \mid \theta)$, and choosing, as a conjugate family, the class of distributions with the same functional form as these likelihood functions.
We demonstrate in the following that the class of normal priors is a conjugate family for the class of normal likelihoods. Suppose that $x_1, x_2, ..., x_n$ are a sample of observations from a population X with an underlying *pdf* $f_x(x)$; the probability of observing that particular set of values assuming that the parameter of the distribution is θ, also called likelihood function $L(\theta)$, is represented by the product of the *pdf* of X evaluated at $x_1, x_2, ..., x_n$:

$$p(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} f_x(x_i|\theta)dx. \tag{3.8}$$

Suppose now that $x_1, x_2, ..., x_n$ are observed from a Gaussian population with known standard deviation σ; then the likelihood function for the parameter μ is:

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right] = \prod_{i=1}^{n} N_\mu\left(x_i, \sigma\right), \tag{3.9}$$

where $N_\mu(x_i, \sigma)$ denotes the *pdf* of μ with mean value x_i and standard deviation σ. According to [6], the product of m normal *pdf*s with respective means μ_i and standard deviations σ_i is also a normal *pdf* with mean:

$$\mu^* = \frac{\sum_{k=1}^{m} \frac{\mu_i}{\sigma_i^2}}{\sum_{k=1}^{m} \frac{1}{\sigma_i^2}}, \tag{3.10}$$

and variance

$$(\sigma^*)^2 = \frac{1}{\sum_{k=1}^{m} \frac{1}{\sigma_i^2}}. \tag{3.11}$$

Therefore the likelihood function $L(\mu)$ becomes:

$$L(\mu) = N_\mu\left(\bar{x}, \frac{\sigma}{\sqrt{n}}\right), \tag{3.12}$$

where $\overline{x}$ is the sample mean. Recalling that $\pi(\mu)$ is the prior distribution characterized by $N_\mu\left(\mu', \sigma'\right)$, the posterior distribution becomes:

$$\pi''(\theta) = kL(\mu)\pi(\mu) = kN_\mu\left(\bar{x}, \frac{\sigma}{\sqrt{n}}\right) N_\mu\left(\mu', \sigma'\right) \tag{3.13}$$

which is the product of two normal *pdf*s. For this reason also $\pi''(\theta)$ is normal with mean

$$\mu'' = \frac{\bar{x}\left(\sigma'\right)^2 + \mu'\left(\frac{\sigma^2}{n}\right)}{\left(\sigma'\right)^2 + \left(\frac{\sigma^2}{n}\right)} \tag{3.14}$$

and standard deviation

$$\sigma'' = \sqrt{\frac{\left(\sigma'\right)^2 \frac{(\sigma)^2}{n}}{\left(\sigma'\right)^2 + \frac{(\sigma)^2}{n}}}. \tag{3.15}$$

There are several different ways of interpreteting the form of the posterior mean μ''. It is also possible to express Eq. 3.16 as a weighted average of the prior mean and the observed value $\bar{x}$, with weight proportional to the inverse of their variances, also called precisions. Another way is to consider μ'' as the prior mean adjusted toward the observed $\bar{x}$ [74]:

$$\mu'' = \mu' + \left(\bar{x} - \mu'\right) \frac{\sigma^2}{\sigma^2 + \frac{\sigma^2}{n}}. \tag{3.16}$$

We notice that Eq. 3.16 is linear in μ'. The Bayes estimator of μ yields:

$$\hat{\mu}'' = \mu''. \tag{3.17}$$

Conjugate distributions introduce several simplications in the computation of the Bayesian estimators and thus posterior distribution, and they are selected because of the mathematical convenience and simplicity. For a random variable with a specified distribution, its conjugate prior distribution may be adopted if there is no other basis for the choice of the prior distribution. Conversely, if evidence to support a particular prior distribution exists, then that distribution should be chosen, mathematical complications notwithstanding [6].

3.4 Computational aspects of the posterior distribution

Unless the prior distribution is conjugated through the likelihood to the posterior, the computation of the posterior distribution and thus of the Bayes estimator involves several integrals and for this reason, it is not straightforward.This Chapter concerns classical techniques that facilitate Bayesian calculation, as well as more recent but very popular procedure according which sample of the posterior distribution can be obtained, namely Markov Chain Monte Carlo (MCMC).

3.4.1 Classical approximations methods

Many approaches have been developed in order to approximate integrals numerically; a procedure that will be also deepened in the Chapter 4 is represented by polynomial quadrature. However, no matter the numerical integration method implemented, its accuracy decreases as the number of random variables increases. An empirical rule of thumb is that most of the standard methods should not be used for integration in dimension larger than

4. Otherwise, if the function to integrate is regular enough, it is possible to apply another method that leads to an analytical formulation and it is based on the Laplace expansion of a general integral.

3.4.2 Markov Chain Monte Carlo

MCMC methods take advantages of asymptotic properties, as well as of the fact that $\pi(\theta)$ is a density and thus $\pi(\theta)L(\theta)$ is proportional to a density. The implementation of these methods requires the production of samples θ_i by a computer, that can be executed according to different strategies [133]. Examples of MCMC algorithms are the Gibbs sampling and the Metropolis-Hastings (MH). The basic idea behind MCMC is to perform a random walk that, if unmodified, would sample some initial *pdf*; then, using a probabilistic rule, the walk is modified in such a way that it samples from the target distribution, represented in this specific case by the posterior *pdf*. In doing so the MH probably represents the most efficient algorithm [159, 162]. Assuming that $\pi(\theta)$ is the prior density and $L(\theta)$ is the likelihood function, the posterior distribution whose samples are sought is $\pi(\theta|x)$ and it is known up to a normalization factor. Starting from an arbitrary value y^0, MH generates a chain updating from y^m to y^{m+1} according to the Algorithm 1.

Algorithm 1 Metropolis-Hastings

Generate $y^* \sim q(y^*|y^m)$
Compute $\rho = \frac{\pi(y^*)L(\theta|y^*)}{\pi(y^m)L(\theta|y^m)}$
if $\rho > 1$ **then**
    $y^{m+1} = y^*$
else if $\rho < 1$ **then**
    sample $r = U[\,0, 1]$
    if $r < \rho$ **then**
        $y^{m+1} = y^*$
    else if $r > \rho$ **then**
        $y^{m+1} = y^m$
    end if
end if

Applying the algorithm, a Markov Chain y^1, y^2, ..., y^m is generated, that converges after a certain burning period to $\pi(\theta|x)$. $q(.)$ represents the so-called jumping or proposal distribution that, in case of MH, it is characterised by a symmetric distribution $q(y^*|y^m) = q(y^m|y^*)$. MH is a very general and reliable procedure; however, the practical implementation is usually affected by slow convergence and thus inefficient estimation. Factors that mainly affect the efficiency of the algorithm are the choice of the initial values and the spread of the proposal distribution. In effect, the algorithm converges slowly if initial values are in a region of low probability. A bad choice of the proposal distribution might lead to two problems: 1) jumps are so short that the simulation moves slowly through the posterior distribution; 2) jumps are nearly all into low probability areas of the posterior distribution, causing the Markov Chain to stand still most of the time. In case of slow simulation, it is possible to improve the mixing by monitoring the frequency of acceptance and thus adjusting the proposal distribution. If the acceptance rate is very close to 1, it suggests that the proposal distribution is too narrow; conversely, if it is close to 0, the proposal distribution is too large. According to [73], the proposal should be scaled in such a way that the average acceptance rate is 1/4. According to [135] the algorithm efficiency is high whenever the acceptance rate is between 0.1 and 0.6.

3.5 Bayesian robustness

A problem that deserves special attention is represented by the robustness of Bayes' rule to the choice of the prior distribution. With 'robustness' it is meant the sensitivity of the results of the Bayesian updating to the prior probability model when it is affected by uncertainty. Although also the sample density might be uncertain, its specification is usually less subjective compared to the prior *pdf*, and for this reason, attention is here focused on its definition. The prior distribution is composed of a central part and the tails. The central part usually represents the 90 % credible interval for the considered parameter, while the tails are constructed upon the extreme regions of small probabilities. The Bayesian updating is usually robust with respect to small changes in the central portion of the prior, but only rarely it is robust with respect to large changes unless the sample information overwhelms the prior information [15]. The tail of the prior is usually affected by great uncertainty, therefore robustness with respect to the tail is desirable. If the likelihood function is concentrated in the central portion of the prior, then the tail will not have a serious impact. Conversely, if data are located at the extremes, the posterior distribution will be affected by the type of prior tail chosen, although this fact might suggest that the prior distribution is wrong.

3.5.1 Defining robust priors

Without entering into the details of how robustness can be measured, attention is here focused on which priors tend to be 'robust'; a robust prior is not only the one with few uncertainties, but also the one that properly models the uncertainty affecting the problem, or in simple words, the 'right' one. By their definition, both the noninformative and maximum likelihood priors described in Chapter 3.3.3.3 represent a robust choice whenever only scarce prior information is available. Although this can be the case of ancient buildings, it has been already noticed in Chapter 3.3.2.1 that, since the beginning of the modern engineering practice, some kind of prior information can be retrieved in most of the cases. A valuable source of prior information is especially represented by the huge amount of data collected in laboratory archives or other types of databases, such as the results of material acceptance tests or measurements of action collected during field studies. Nonetheless, two problems here arise: how is it possible to use this information in order to construct better prior distribution? And how the engineer can select the 'right' one? Those problems especially arise when a prior distribution for a material mechanical parameter has to be selected since materials are usually classified according to resistance classes. In order to solve those problems, also Bayesian empirical priors described in Chapter 3.3.3.4 and 3.3.3.4 could be considered and combined with other procedures that will make the choice of the prior distribution less subjective.

3.6 Summary

This chapter introduces the cornerstone subject of this work, that is the application of Bayes' theorem for the definition of probability models. First of all, the basic concepts of uncertainty, random variables, population and sample are reviewed in the framework of the reliability assessment of existing structures. Then Bayes' theorem is described and the significance of the Bayesian updating is illustrated considering classical probability and Bayesian statistics. Attention is drawn on the procedures which allow the definition of the prior distribution, especially those that in this specific framework can increase the robustness of Bayes' rule, like empirical Bayes methods. Finally, the computational aspects of the Bayesian analysis are tackled, with a special emphasis on the most general and reliable MCMC method.

Chapter 4

Bayesian approach to the stochastic inverse problem

The general polynomial chaos expansion is a method for quantifying the uncertainty on the output of a system given uncertain input parameters, based on a representation of the random variables in mathematical series forms. Such representation has several advantages: among the other, it facilitates the identification procedure when measurements of the model response are available. In this chapter, it is shown how classical updating procedures can be adapted to deal with the functional approximation of the system response. The performance of those methods and the impact of the elements involved in the updating process is finally investigated considering simple univariate analytical models.

Contents

4.1 Introduction

Let us consider a mechanical system whose behavior is modeled by a set of governing equation, i.e. partial differential equations. Let us suppose that the mechanical model is characterized by a vector $\mathbf{Z} \in \mathbb{R}^n$ of input random parameters, each Z_i described by a prior *pdf* $\pi_i(z_i)$. Assuming mutually independent variables, the joint prior density function for $\mathbf{Z}$ is [171]:

$$\pi_{\mathbf{Z}}(\mathbf{z}) = \prod_{i=1}^{n} \pi_i(z_i). \tag{4.1}$$

Let the relationship between the vector $\mathbf{Z}$ and the observable $\mathbf{u}$ given by the forward model G:

$$\mathbf{u} = G(\mathbf{Z}), \tag{4.2}$$

where $\mathbf{Z} \in \mathbb{R}^m$ is a vector that gathers the response quantities and $G : \mathbb{R}^n \to \mathbb{R}^m$. It is assumed here that the computational model is a deterministic black box: selecting a given set of input parameters $\mathbf{Z}$, running the model provides a unique response vector $\mathbf{u}$, that will not change if the model is run again with the same input. Since measurement errors are inevitable in practice, observable data $\mathbf{d}$ may not match the model response $\mathbf{u}$. Assuming additional observational errors $\boldsymbol{\varepsilon}$, it results:

$$\mathbf{d} = \mathbf{u} + \boldsymbol{\varepsilon} = G(\mathbf{Z}) + \boldsymbol{\varepsilon}, \tag{4.3}$$

where $\boldsymbol{\varepsilon} \in \mathbb{R}^m$ are mutually independent random variables, supposed independent on $\mathbf{Z}$, with probability density functions:

$$\pi(\boldsymbol{\varepsilon}) = \prod_{i=1}^{m} \pi(\varepsilon_i). \tag{4.4}$$

The Bayesian approach seeks to estimate the random vector $\mathbf{Z}$ given a set of observations $\mathbf{d}$. Bayes' rule takes here the following form:

$$\pi(\mathbf{z}|\mathbf{d}) = \frac{\pi(\mathbf{d}|\mathbf{z})\pi(\mathbf{z})}{\int \pi(\mathbf{d}|\mathbf{z})\pi(\mathbf{z})dz}, \tag{4.5}$$

where $\pi(\mathbf{z})$ is the prior probability density of $\mathbf{Z}$, $\pi(\mathbf{d}|\mathbf{z})$ is the likelihood function, and $\pi(\mathbf{z}|\mathbf{d})$ is the density of $\mathbf{Z}$ conditioned by the data $\mathbf{d}$, or in other words the posterior probability density of $\mathbf{Z}$. In contrast to the direct

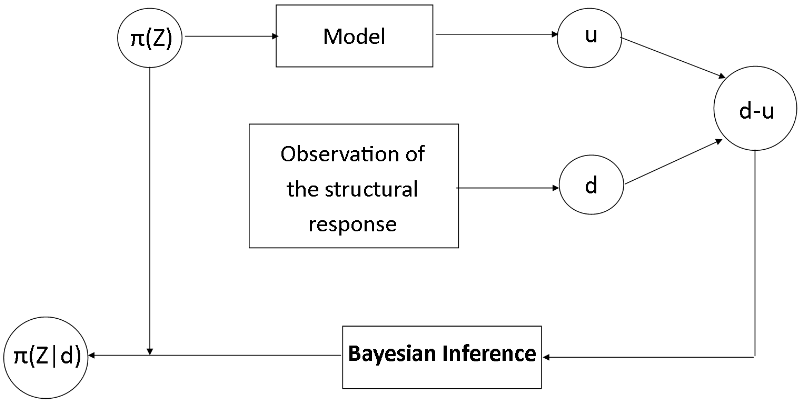

Figure 4.1: Schematic representation of the Bayesian approach to the stochastic inverse problem.

Bayesian analysis, the end product is the posterior distribution of $\mathbf{Z}$, and not the predictive distribution of $\mathbf{d}$; the statistical parameters of the random vector $\mathbf{Z}$ play the role of the "hyperparameters" of the distribution of $\mathbf{d}$ [156]. Eq. 4.5 implies that $\mathbf{Z}$ and $\mathbf{d}$ have a joint *pdf*; however $\mathbf{d}$ is a function of $\mathbf{Z}$ and therefore a joint density does not generally exist. A possibility when a joint density may be established is when the observational error is a discrete white noise process. In this case the model for the random variable representing the error $\boldsymbol{\varepsilon}$ determines the existence of the likelihood function:

$$L(\mathbf{z}) = \pi(\mathbf{d}|\mathbf{z}) = \prod_{i=1}^{m} \pi_{\varepsilon_i}(d_i - G_i(z)) = \prod_{i=1}^{m} \pi_{\varepsilon_i}(d_i - u_i). \tag{4.6}$$

An increasing amount of literature is devoted to the computational aspects of the Bayesian approach to the stochastic inverse problem. In this work the attention is mainly focused on those methods that take advantage of functional approximations of the random variables through general Polynomial Chaos Expansion (gPCE). In Chapter 4.2 the gPCE is introduced and in Chapter 4.3 it is shown how MCMC and the methods based on the Kalman Filter (KF) and the Minimum Mean Squared Error (MMSE) estimator can be adapted to deal with this representation.

Distribution of Z	gPC basis polynomials	Support
Gaussian	Hermite	$(-\infty, +\infty)$
Gamma	Laguerre	$[0, +\infty)$
Beta	Jacobi	$[a, b]$
Uniform	Legendre	$[a, b]$

Table 4.1: Correspondence between the type of Generalized Polynomial Chaos and their underlying random variables [171].

4.2 Solution to the forward problem: general Polynomial Chaos Expansion

The gPCE is a method for quantifying the uncertainty in the output of a system given uncertain input parameters, based on a representation of the random variables in mathematical series forms. Let us consider at first the case in which Z is a single random variable having *pdf* $\pi_Z(z)$ and finite moments. The generalised polynomial chaos basis functions are the orthogonal polynomials satisfying [171]:

$$\mathbb{E}\left[\phi_i(Z)\phi_j(Z)\right] = \gamma_i \delta_{ij}, i, j \in N_0 \tag{4.7}$$

where

$$\gamma_i = \mathbb{E}\left[\phi_i(Z)^2\right], i \in N_0 \tag{4.8}$$

is the normalization factor. If Z is continuous, then the orthogonality can be written as:

$$\mathbb{E}\left[\phi_i(Z)\phi_j(Z)\right] = \int \phi_i(z)\phi_j(z)\pi_Z(z)dz = \gamma_i \delta_{ij}, i, j \in N_0. \tag{4.9}$$

$\{\phi_i(z)\}$ are orthogonal polynomials of $z \in \mathbb{R}$ with the weight function $\pi_Z(z)$ which is the *pdf* of the random variable Z. This establishes a correspondence between the distribution of the random vector Z and the type of orthogonal polynomials of its gPCE basis (Table 4.1).

Assume here to be in the multivariate case, having a random vector $\mathbf{Z} =$

$(Z_1, ..., Z_d)$ with finite moments. $\{\phi_k(Z_i)\} \in \mathbb{P}_N(Z_i), k = 0, ..., N$ are the univariate basis functions in Z_i that satisfies the condition expresses by Eq. 4.9. Let $\mathbf{i} = (i_1, ..., i_d) \in N_0^d$ is a multi-index with $|\mathbf{i}| = i_1 + ... + i_d$. Then the d-variate Nth degree gPC basis functions are the products of the univariate gPCE polynomials of total degree less or equal to N, i.e.:

$$\phi_{\mathbf{i}}(\mathbf{Z}) = \phi_{i_1}(Z_1)...\phi_{i_d}(Z_d), 0 < |\mathbf{i}| < N. \tag{4.10}$$

It follows immediately form Eq. 4.9 that:

$$\mathbb{E}\left[\phi_{\mathbf{i}}(\mathbf{Z})\phi_{\mathbf{j}}(\mathbf{Z})\right] = \int \phi_{\mathbf{i}}(z)\phi_{\mathbf{j}}(z)\pi_Z(z)dz = \gamma_{\mathbf{i}}\delta_{\mathbf{ij}}, \tag{4.11}$$

where $\delta_{\mathbf{ij}} = \delta_{i_1 j_1} \dots \delta_{i_d j_d}$ is the d-variate Kronecker delta function and

$$\gamma_{\mathbf{i}} = \mathbb{E}\left[\phi_i(\mathbf{Z})^2\right] = \gamma_{i_1}...\gamma_{i_d} \tag{4.12}$$

are normalization factors.
Consider now the response quantity of the previously considered structural system $G(\mathbf{Z})$; then its expansion is represented by:

$$G(\mathbf{Z}) = \sum_{\mathbf{i} \in N_0^d} \hat{u}_i \phi_i(\mathbf{Z}), \tag{4.13}$$

where $\hat{u}_i$ are the coefficients that have to be computed. Once that the basis is built, a troncature scheme has to be selected in order to carry out the computation of the coefficients. Usually all the polynomials of total degree $|\mathbf{i}|$ not greater than a certain N are selected. Then the Nth-degree gPCE approximation of $\mathbf{u} = G(\mathbf{Z})$ is:

$$G_N(\mathbf{Z}) = \sum_{|\mathbf{i}| \leq N} \hat{u}_i \phi_i(\mathbf{Z}). \tag{4.14}$$

The computation of the coefficients can be carried out according to different methods such as interpolation/regression (also called collocation), projection, and Galerkin. A brief outlook of the interpolation/regression and projection methods is given in the following, since those procedures have been effectively applied in the present work. For an comprehensive review about this topic, please refer to [163].

Interpolation/regression According to this method, the coefficients of the functional approximation are obtained by calculating the model for certain input values $(\xi_\alpha)_{\alpha=1}^M$, where M is the number of points. Let us consider for simplicity an univariate model $u = G(Z)$, whose approximation with polynomials of Nth order is $u_N = G_N(Z)$. Then $M = N+1$ (if $M > N+1$, then the coefficients are defined through regression). The coefficients can be calculated equaling the proxy model to the calculated points:

$$\sum_{|\mathbf{i}|\leq N} \hat{u}_i \phi_i(\xi_\alpha) = G(\xi_\alpha) \quad \alpha = 1...M. \tag{4.15}$$

The same equation can also be written in matrix form:

$$\begin{bmatrix} \phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_N(\xi_1) \\ \phi_0(\xi_2) & \phi_1(\xi_2) & \cdots & \phi_N(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_N(\xi_M) \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix} = \begin{bmatrix} G(\xi_1) \\ G(\xi_2) \\ \vdots \\ G(\xi_M) \end{bmatrix} \tag{4.16}$$

The problem is well defined if the interpolation matrix is not singular. A further problem may arise when the number of input random variables is high. Given that r is the number of random variables, the model has to be evaluated M^r times. Therefore the number of points for which the model has to be evaluated increases exponentially with the stochastic dimension. The curse-of-dimensionality problem can be partially solved by resorting to the projection method.

Projection Coefficients can also be computed orthogonally projecting the response surface to the subspace $u_N = \text{span}\{\phi_i\}$. The projection assures that in a given norm the error is minimised in that subspace u_N:

$$\sqrt{\langle (u(Z) - u_N(Z)), (u(Z) - u_N(Z)) \rangle} = min. \tag{4.17}$$

The error attains its minima when it is orthogonal to the approximating subspace u_N; using the expected value for the norm we obtain:

$$\langle (u(Z) - u_N(Z)), \phi_j(Z)) \rangle = \mathbb{E}\left[(u(Z) - u_N(Z))\phi_j(Z) \right], j = 0...N. \tag{4.18}$$

Rearranging the equation and using the orthogonality condition in equation 4.9 we obtain:

$$\langle (u(Z) - \sum_{i \leq N} \hat{u}_i \phi_i(Z)), \phi_j(Z)) \rangle = 0, j = 0...N, \tag{4.19}$$

$$\langle (u(Z) - \phi_i(Z)) \rangle = \sum_{i \leq N} \hat{u}_i \langle \phi_i(Z) \phi_j(Z) \rangle, j = 0...N, \tag{4.20}$$

where $\langle \phi_i(Z) \phi_j(Z) \rangle = \gamma_i \delta_{ij}$.
The coefficients can be thus computed in the following way:

$$\hat{u}_i = \frac{1}{\gamma_i} \langle (u(Z), \phi_j(Z)) \rangle = \frac{1}{\gamma_i} \mathbb{E} \left[(u(Z) \phi_j(Z) \right] = \int_{-\infty}^{\infty} u(Z) \phi_j(Z) \pi_Z(z) dz. \tag{4.21}$$

Unfortunately, the integral can not be directly calculated, as the dependence between u and Z is not known. However it can be evaluated by a quadrature rule:

$$\hat{u}_i \approx \sum_{\alpha=1}^{M} u(\xi_\alpha) \phi_j(\xi_\alpha) w_\alpha, \tag{4.22}$$

where the $u(\xi_\alpha)$ values have to be evaluated by the deterministic solver.

4.3 Solution to the inverse problem

4.3.1 Markov Chain Monte Carlo

In order to calculate the acceptance rate, the likelihood function of Eq.4.6 must be computed, that in turn results in the simulation of the model for each new sample drawn from the prior distribution of the input parameter. For this reason, MCMC algorithm is a very demanding procedure, as the system response must be evaluated for each newly proposed sample. In order to improve the efficiency of the MCMC method, the functional approximation of the random variables entering into the model can be used. In effect, one can directly sample from the general polynomial chaos expansion approximation of the parameters of $\mathbf{Z}$ and the measured response u, instead of solving the system for all the samples [136]. Then the approximated likelihood function is defined as:

$$L_N(\mathbf{z}) = \pi_N(\mathbf{d}|\mathbf{z}) = \prod_{i=1}^{m} \pi_{\varepsilon_i}(d_i - G_{N,i}(z)) = \prod_{i=1}^{m} \pi_{\varepsilon_i}(d_i - u_{N,i}) \tag{4.23}$$

and the PCE approximation of the posterior probability is:

$$\pi_N(\mathbf{z}|\mathbf{d}) = \frac{L_N(\mathbf{z})\pi(\mathbf{z})}{\int L_N(\mathbf{z})\pi(\mathbf{z})dz}. \tag{4.24}$$

4.3.2 Linear Bayesian Filter

4.3.2.1 Kalman Filter and Ensemble Kalman Filter

The KF is a method for sequential state estimation for incompletely observable, linear discrete-time dynamics; it consists of two stages: 1) a forecast stage when the system of interest is solved and the forecast solution is obtained; 2) an analysis stage when the forecast stage and the data are combined in order to obtain better prediction of the system [94]. The KF scheme can be also applied to solve inverse problems as that one expressed by Eq. 4.2. The analysed solution $\mathbf{z}^a$ is determined as a combination of the forecast solution $\mathbf{z}^f$ and the measurement $\mathbf{d}$ in the following manner:

$$\mathbf{z}^a = \mathbf{z}^f + K(\mathbf{d} - G(\mathbf{z}^f)) = \mathbf{z}^f + K(\mathbf{d} - \mathbf{u}^f), \tag{4.25}$$

where K is the Kalman gain:

$$K = Cov(\mathbf{z}^f, \mathbf{u}^f)[Cov(\mathbf{u}^f) + Cov(\boldsymbol{\varepsilon})]^{-1}, \tag{4.26}$$

which can be easily evaluated with the covariance matrices $Cov(\mathbf{z}^f, \mathbf{u}^f)$, $Cov(\mathbf{z}^f)$ and $Cov(\boldsymbol{\varepsilon})$. Please notice that for the trivial case $u = Z$ and considering a normal random variable with the prior mean equals to z^f and the posterior mean equals to z^a, Eq. 4.25 reduces to Eq. 3.16.
The covariance function $Cov(\mathbf{z}^a)$ of the analysed state $\mathbf{z}^a$ is then obtained by:

$$Cov(\mathbf{z}^a) = Cov(\mathbf{z}^f) - KCov(\mathbf{z}^f, \mathbf{u}^f). \tag{4.27}$$

However, if the system is not linear, an explicit derivation of the covariance function is not possible. Subsequently, various approximations have been developed. Among the others, the Ensemble Kalman Filter (EnKF) overcomes the limitation of the KF by using an ensemble approximation of the random state solution [62]. Let

$$(\mathbf{z}^f)_i, i = 1, ..., M, M > 1 \tag{4.28}$$

be an ensemble of the forecast solution $\mathbf{z}^f$, where each ensemble member is indexed in the subscript $i = 1, ..., M$ and it is obtained by solving the full nonlinear system. The analysis step for the EnKF consists of the following update performed on each of the model state ensemble members:

$$(\mathbf{z}^a)_i = (\mathbf{z}^f)_i + K_e((\mathbf{d})_i - G(\mathbf{z}^f)_i), i = 1, ..., M, \tag{4.29}$$

where

$$K_e = Cov(\mathbf{z}^f, \mathbf{u}^f)\left[(\mathbf{u}^f)_e(\boldsymbol{\varepsilon})_e\right]^{-1} \tag{4.30}$$

is the ensemble Kalman gain. Here

$$Cov(\mathbf{z}^f)_e = \overline{(\mathbf{z}^f - \overline{\mathbf{z}^f})(\mathbf{z}^f - \overline{\mathbf{z}^f})^T} \tag{4.31}$$

$$Cov(\mathbf{z}^a)_e = \overline{(\mathbf{z}^a - \overline{\mathbf{a}^f})(\mathbf{a}^f - \overline{\mathbf{a}^f})^T} \tag{4.32}$$

are the approximate forecast covariance and analysis covariance, respectively, obtained by using statistical averages of the solution ensemble, and $Cov(\boldsymbol{\varepsilon})_e = \overline{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T} \cong Cov(\boldsymbol{\varepsilon})$ is the approximate observation error covariance.

4.3.2.2 Improved EnKF with gPCE

To avoid the sampling procedure required by the EnKF, one may resort again to a functional approximation of the random variables; in this way, the linear Bayesian procedure is reduced to a simple algebraic method. To this end, both the prior and the predicted system response can be represented in a polynomial chaos expansion form:

$$z_N^f(\mathbf{Z}) = \sum_{\mathbf{i} \leq N} \hat{z}_i^f \phi_i(\mathbf{Z}), \tag{4.33}$$

$$u_N^f(\mathbf{Z}) = \sum_{\mathbf{i} \leq N} \hat{u}_i^f \phi_i(\mathbf{Z}), \tag{4.34}$$

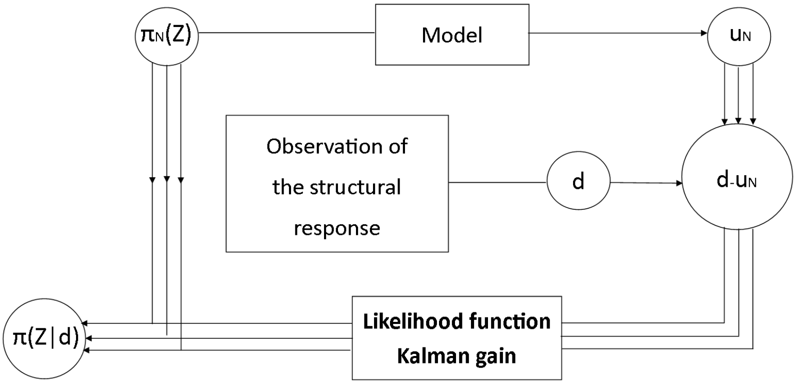

Figure 4.2: Schematic representation of the Bayesian approach to the stochastic inverse problem also considering a functional approximation of the system response u_N. In order to compute the Kalman gain or the likelihood function, it is possible to draw samples directly from the response surface. This prevents the simulation of the model for a huge number of times.

where $\phi_i(\mathbf{Z})$ are the generalised orthogonal polynomials and $\mathbf{i}$ is the multi-index; then it possible to discretise Eq. 4.25 in the following way:

$$\mathbf{z}_N^a = \mathbf{z}_N^f + K(\mathbf{d}_N - \mathbf{u}_N^f). \tag{4.35}$$

Here, K is the Kalman gain evaluated in an algebraic way knowing that:

$$Cov(\mathbf{z}^f, \mathbf{u}^f) = \sum_{N>0} N! \mathbf{z}_N^f (\mathbf{u}_N^f)^T. \tag{4.36}$$

4.3.3 Method based on the Minimum Mean Squared Error estimator

It is possible to tackle the problem of the Bayesian updating exploiting the properties of the MMSE estimator [17]. Recalling that $\mathbf{Z} \in \mathbb{R}^n$ is the vector of input random parameters and $\mathbf{d} \in \mathbb{R}^m$ is the measurement of the system output, an estimator $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is any function of the measurement $\mathbf{d}$. According to [13], it is possible to demonstrate that

$$\hat{\varphi} = \mathbb{E}\left[\mathbf{Z}|\mathbf{d}\right] \tag{4.37}$$

is the estimator that minimizes the conditional mean squared error

$$e^2_{MSE} = \mathbb{E}\left[(\varphi - \mathbf{Z})^2|\mathbf{d}\right]. \tag{4.38}$$

$\hat{\varphi}$ represents the minimum mean squared error estimate of $\mathbf{Z}$ given $\mathbf{d}$ and it assumes a particular importance in the framework of Bayesian inference because it makes suitable nonlinear Bayesian updating [14]. In order to carry out the minimization, φ is defined over the space of finite dimensional function V_φ with basis function ψ_i that could be some sort of multivariate polynomials with $\mathbf{i}$ the corresponding multi-index. An element φ of this function space has a representation as a linear combination of these basis functions up to degree P [14]:

$$\varphi := \mathbf{d} \mapsto \sum_{|\mathbf{i}|<P} \varphi_{\mathbf{i}} \psi_{\mathbf{i}}(\mathbf{d}). \tag{4.39}$$

It is important to notice here that φ is not actually a gPCE because it does not represent a random quantity; it rather represents a multivariate polynomial which shares many properties with the gPCE variables. Notice also that if $P = 1$ and considering again the trivial case $u = Z$ with Z described by a Gaussian *pdf*, then the posterior mean is obtained again like in Eq. 3.14. For the detailed formula that leads to this result please refer to [6]. For further details regarding the problem of computing the minimiser, please refer to [175].

4.3.4 Comments

The MCMC algorithm represents a general approach that leads to trustworthy results in most of the cases, but it presents a slow convergence compared to the Polynomial Chaos Expansion based Kalman Filter (PCE-KF) and the MMSE methods, although an analytical representation of $\mathbf{u}$ in terms of $\mathbf{Z}$ is available. The PCE-KF represents the linearized version of the MMSE: this means that the MMSE leads to the same results of PCE-KF if the degree P of polynomials used for the approximations is equal 1. In general, the EnKF leads to the correct conditional expectation, while the conditional mean could be only approximated. The correct conditional mean can be given only if $\mathbf{Z}$ is a vector of independent Gaussian random variables and $\boldsymbol{\varepsilon}$ is a vector of independent Gaussian noises [112, 60].

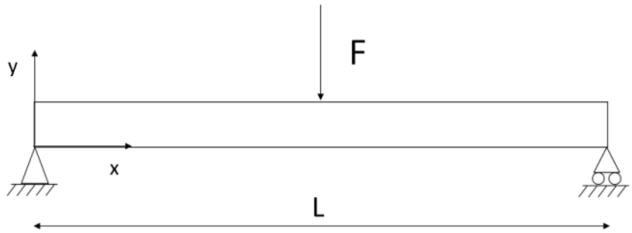

Figure 4.3: Simply supported bending beam submitted to mid-span loading.

4.4 Examples

4.4.1 Solution of the forward problem for a simple analytical model

To illustrate the operational application of PCE based method in solving the forward problem, a simple case study is considered here. The example deals with a simply supported reinforced concrete beam, loaded by a concentrated force F at midspan (Fig. 4.3).

4.4.1.1 Mechanical model statement

In an abstract form this system can be defined by:

$$\mathcal{A}\left[u(x), \mathbf{Z}\right] = F, \tag{4.40}$$

where $\mathcal{A}$ is the operator that defines the relation between $u(x)$, the solution of the equation or model response, represented by the displacement of the beam under the force F, and $\mathbf{Z}$, the vector of input parameters, that comprises geometric quantities (Table 4.2) and the concrete elastic modulus E_c. According to [56], E_c can be related to the concrete resistance through

$$E_c = 22\left[\frac{f_{cm}}{10}\right]^{0.3}, \tag{4.41}$$

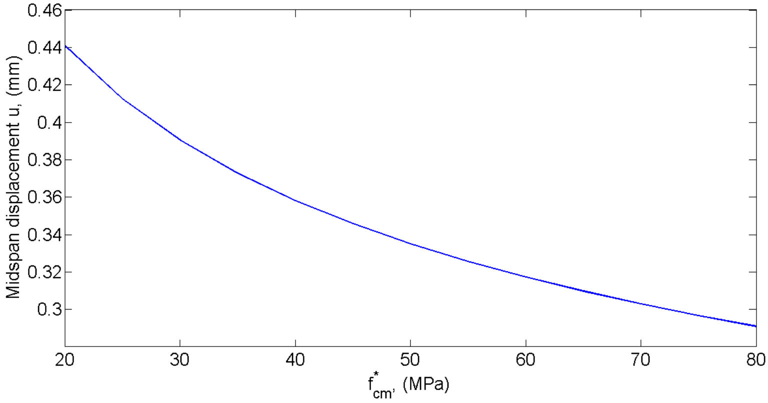

Figure 4.4: Non-linear relationship between the response of the structure and f^*_{cm}.

where f_{cm} is the mean value of the concrete strength f_c. Since the influence of other parameters, like concrete mix, cement type, water-cement ratio, type and dimensions of aggregate and so on, cannot be disregarded, in reality the $E_c - f_{cm}$ plots are very scattered, so that the dependency of E_c on f_{cm} expressed by 4.41 results very rough. A simple way to take into account this phenomenon, adjusting the values f_c in order to consider also actual scattering of E_c, is to introduce in Eq. 4.41, instead of f_{cm}, the term $f^\star_{cm}$, which is the mean value of the random variable $f^\star_c = (f_c + \Delta f)$, where Δf is a suitable random correction, uncorrelated to f_c, and characterized by a symmetrical distribution with zero mean. Thus the forward model that describes the relationship between $u(L/2)$ and f_{cm} and predicts the observation is given by:

$$u(x = L/2) = \frac{FL^3}{48E_cI} = \frac{FL^3}{48\left[22\left[(f_{cm} + \Delta f_m)/10\right]^{0.3}\right]I} = G(f_{cm} + \Delta f), \tag{4.42}$$

where F is the force located at the midspan, L is the lenght of the beam and I is the section moment of inertia. L and the values of the parameters that lead to determine I are given in Table 4.2. In Fig. 4.4, it is diagrammatically represented the $u - f^\star_{cm}$ curve as $\Delta f = 0$.

Reinforcement area	Effective height	Width	Length
$A_s[mm^2]$	$d[mm]$	$b[mm]$	$L[m]$
804 (4 ϕ 16)	255	600	3

Table 4.2: Geometric properties of the reinforced concrete beam.

4.4.1.2 Probabilistic models, computational algorithm and results

In the example, it is assumed that the distribution of concrete strength is a normal distribution characterized by a mean value $\mu_{f_c} = 50\ MPa$. Hypothesizing that the concrete production is well controlled, a coefficient of variation (COV) of 0.14 can be adopted for the concrete resistance, so that we obtain a standard deviation $\sigma_{f_c} = 7\ MPa$. Regarding Δf, a normal distribution is assumed with a standard deviation $\sigma_{\Delta f} = \sigma_{f_c} = 0.14\mu_{f_c}$, so that in the present case $\sigma_{\Delta f} = 7\ MPa$. Obviously, being f_c and Δf uncorrelated, $f_c^\star$ is normally distributed and characterized by a mean value $f_{cm}^\star = 50\ MPa$ and a standard deviation of $\sigma_{f_c^\star} = 9.9\ MPa$ ($COV = 0.198$).
The following algorithm has been applied in order to solve the forward problem:

1. Map $f_c^\star = N(50, 9.9)$ to $\xi = N(0,1)$:

$$f_c^\star = f_c^\star(\xi) = \mu_{f_c^\star} + \sigma_{f_c^\star}\xi. \tag{4.43}$$

   it is possible to write this mapping with the help of the univariate, first order PCE:

$$f_c^\star = \sum_i \hat{z}_i(\xi)H_i(\xi), i = 0, 1 \tag{4.44}$$

   where H_i are Hermite polynomials associated to ξ.

2. Solve the forward problem:

$$u_N = \sum_i \hat{u}_i(\xi)H_i(\xi), i = 0, 1, 2, 3, 4. \tag{4.45}$$

   The response u is expanded until the third order, therefore in this case $N = 3$ and the considered Hermite polynomials are: $H_0(\xi) = 1$, $H_1(\xi) = \xi$, $H_2(\xi) = \xi^2 - 1$, $H_3(\xi) = \xi^3 - 3\xi$. The coefficients are

α	ξ_α	w_α	$u(\xi_\alpha)$
1	-2.334	0.046	0.404
2	-0.742	0.454	0.351
3	2.334	0.454	0.321
4	0.742	0.046	0.299

Table 4.3: Integration points, associated weights and values of the system response.

evaluated through direct projection [163]:

$$\hat{u}_i = \int_{-\infty}^{\infty} u(\xi) H_i(\xi) f_\xi d\xi. \tag{4.46}$$

In general the integral cannot be directly calculated ($u(\xi)$ is not known explicitly). Though here its algebraic form could be calculated, we proceed with presenting the general way. Accordingly, the integral is approximated through a quadrature rule:

$$\hat{u}_i \approx \sum_{\alpha=1}^{4} u(\xi_\alpha) H_i(x_{i_\alpha}) w_\alpha, \tag{4.47}$$

where (ξ_α) are the integration points, w_α are the associated weights and $u(\xi_\alpha)$ is the response calculated considering different $f_c(\xi_\alpha)$ values deterministically evaluated at the integration points (Table 4.3). The following coefficients are obtained: $u_0 = 0.3379$; $u_1 = -0.0215$; $u_2 = 0.0044$;$u_3 = -0.0012$.

Given the values of the coefficients, it is also possible to calculate the statistical parameters of the model response [171]:$\mu_u = 0.338\ mm$ and $\sigma_u = 0.022\ mm$.

4.4.2 A comparative study of stochastic inverse methods

Several factors are involved in the Bayesian approach to the stochastic inverse method, which could influence the results. Generally speaking, the choice of the prior distribution, the non-linearity of the model, the measurement error and the data collected play often a crucial role. Some aspects concern the

functional approximation of the response, such as the degree of gPCE, the choice of the polynomials, and the number of random variables involved. Other aspects concern the algorithm implemented to perform the updating: in case of MCMC, the choice of the initial values and the proposal distribution; in case of the method based on the MMSE estimator, the choice of the approximation subspace for the estimator.
Therefore it is worth to perform simple analyses concerning the aspects that are more crucial in engineering application, such as the linearity/non-linearity of the model, the measurement error, and the choice of the prior distribution; the results of this study are aimed at supporting intuitive answers to very common questions regarding the above-mentioned issues.
Before entering into the merits of these analyses, it is recalled here that the methods applied are the followings: MCMC, PCE-KF and Non-Linear Minimum Mean Squared Error (NL-MMSE) estimator (the linear MMSE is here disregarded because it basically corresponds to the PCE-KF). The parameters regarding each method are set up in the following way: for MCMC, the starting points of the random walk correspond to the mean of the prior distribution, while the variance of the proposal is calibrated step by step in order to improve the efficiency of the algorithm; the posterior distribution is sampled 10000 times, after a burning period of 1000 steps. For the NL-MMSE, $P = 4$.

4.4.2.1 Problem statement

Three toy examples, characterized by an increasing degree of non-linearity, have been considered (Fig. 4.5):

1. $y(x) = 1 + 0.5x$;
2. $y(x) = 1 + 0.5x^2$;
3. $y(x) = 1 + 0.5x^4$.

Given that the input parameter has a normal distribution $\mu = 1.5$ and $\sigma = 0.5$, four different measurements d are considered, each one obtained by setting the true value x_{true} of the input random variable at $\mu + 0.6\sigma$, $\mu + 1\sigma$, $\mu + 2\sigma$ and $\mu + 3\sigma$. The measurement error is generated considering different magnitude of σ_ε (σ_ε equals to 1/10, 1/100 and 1/1000 of $y(x_{true})$). The input random variable has been approximated by Hermite polynomials of $4th$ degree, in order to obtain negligible approximation error with all the considered models.

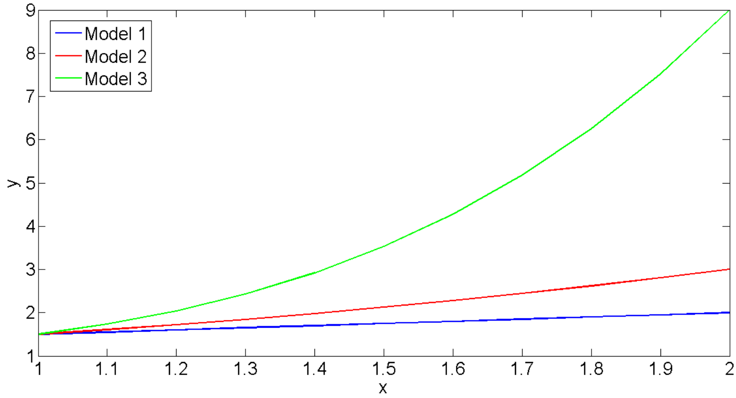

Figure 4.5: Model 1, 2 and 3.

4.4.2.2 Impact of the non-linearity of the model on the posterior distribution

First of all the impact of the non-linearity of the model on the result of the updating is analyzed. For these analyses the true value is fixed at $\mu + 1\sigma$ and σ_ε corresponding to $y(x)/100$. The statistical parameters characterizing the posterior distribution are reported in Table 4.4. If the relationship between inputs and outputs is linear (Model 1), all the methods lead to good results; the posterior distribution is still normal (Fig. 4.6), concentrated around the true value. If the model is slightly non-linear (Model 2), results are still satisfactory: however, when the PCE-KF is applied, the posterior distribution is characterized by a longer tail on the left, and therefore it is not Gaussian anymore (Fig. 4.7). The skewness of the posterior distribution is exacerbated considering the Model 3 (Fig. 4.8); also the numerical results are here less satisfying compared to the previous cases. In this case MCMC appears to be extremely inefficient: whatever the choice of the proposal distribution, the acceptance rate is always very low.

4.4.2.3 Impact of the measurement error on the posterior distribution

The impact of the measurement error has been investigated considering only the Model 2 and simulating the measurement assuming the true value located

True value: $\mu + 1\sigma = 2$, $\sigma_\varepsilon = y(x)/100$								
Model	PCE-KF		MCMC				NL-MMSE	
	μ''	σ''	μ''	σ''	acc. rate	σ^2_{prop}	μ''	σ''
1	2.04	0.04	1.97	0.04	0.30	$\sigma^2/10$	/	/
2	1.97	0.12	2.01	0.02	0.29	$\sigma^2/50$	1.99	0.04
3	1.81	0.25	/	/	/	/	2.06	0.14

Table 4.4: Posterior mean value and standard deviation applying PCE-KF, MCMC, NL-MMSE and considering Model 1,2 and 3.

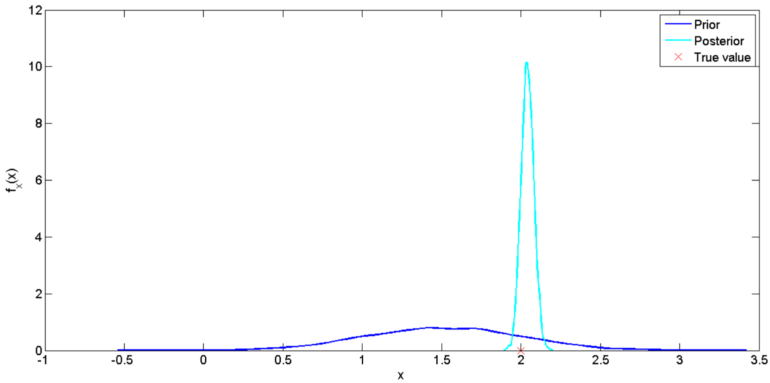

Figure 4.6: Posterior *pdf* obtained applying PCE-KF and considering Model 1.

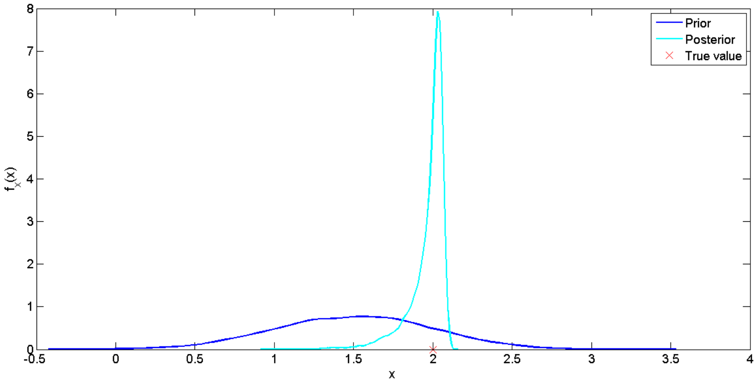

Figure 4.7: Posterior *pdf* obtained applying PCE-KF and considering Model 2.

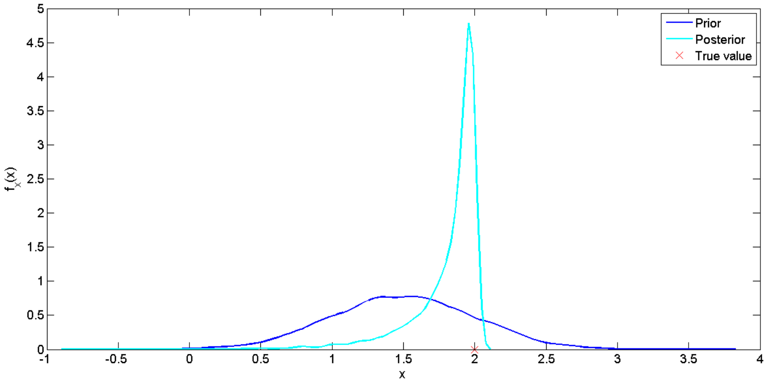

Figure 4.8: Posterior *pdf* obtained applying PCE-KF and considering Model 3.

True value: $\mu + 0.6\sigma = 1.8$, Model 2						
σ_ε	MCMC				NL-MMSE	
	μ''	σ''	acc. rate	σ^2_{prop}	μ''	σ''
y(x)/10	1.85	0.14	0.32	σ^2	1.87	0.18
y(x)/100	1.80	0.014	0.33	$\sigma^2/100$	1.78	0.035
y(x)/1000	/	/	/	/	1.81	0.028

Table 4.5: Posterior mean value and standard deviation applying MCMC and NL-MMSE, considering Model 2 and varying the measurement error.

at 0.6σ (Table 4.5). MCMC and NL-MMSE have been applied. As the measurement error decreases, it is easier to identify the value of the input parameter: the posterior distribution concentrates around that value, and the standard deviation decreases (Fig. 4.9, 4.10 and 4.11). In order to run MCMC efficiently and obtain a good acceptance rate, the proposal distribution has been properly adjusted (the spread has also to be reduced for lower measurement error). However, in case of $y(x)/1000$, the acceptance rate remains very low. In this case the algorithm does not perform efficiently because of the low values of the likelihood function.

4.4.2.4 Distance of the true value from the mean value of the prior distribution

Finally, the impact of the distance of the true value from the mean of the prior distribution has been investigated, considering the Model 2, and fixing the measurement error at $y(x)/100$. Results reveal that the NL-MMSE succeeds in identifying the parameter, although the true value is rather far from the mean of the prior distribution ($\mu + 2\sigma = 2.5$ and $\mu + 3\sigma = 3$, Fig. 4.12 and 4.13). The MCMC performs very badly indeed, regardless the spread of the proposal distribution. This happens because the probability of acceptance is always very low, due to the low values of the prior distribution.

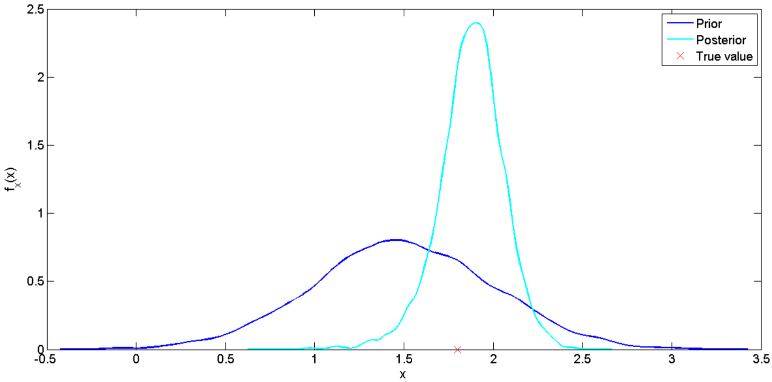

Figure 4.9: Posterior *pdf* obtained applying NL-MMSE and considering $\sigma_\varepsilon = y(x)/10$.

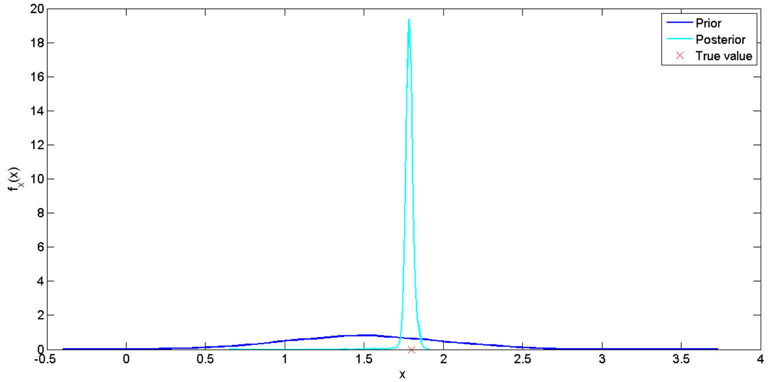

Figure 4.10: Posterior *pdf* obtained applying NL-MMSE and considering $\sigma_\varepsilon = y(x)/100$.

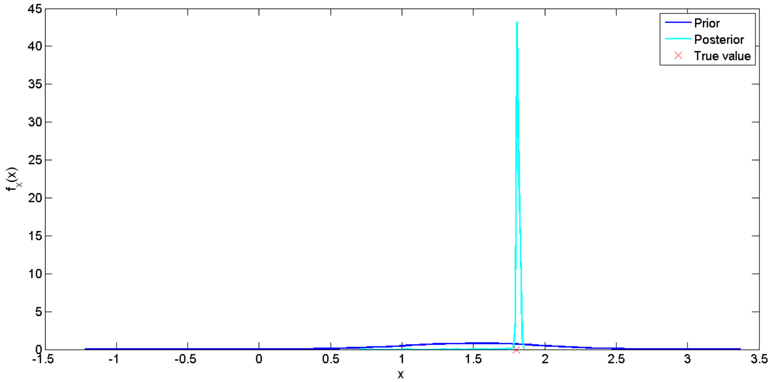

Figure 4.11: Posterior *pdf* obtained applying NL-MMSE and considering $\sigma_\varepsilon = y(x)/1000$.

Model 2, $\sigma_\varepsilon = y(x)/100$						
True value	MCMC				NL-MMSE	
	μ''	σ''	acc. rate	σ^2_{prop}	μ''	σ''
$\mu + 2\sigma$	/	/	/	/	2.48	0.043
$\mu + 3\sigma$	/	/	/	/	3.06	0.052

Table 4.6: Posterior mean value and standard deviation applying MCMC and NL-MMSE, considering Model 2 and varying the distance of the true value from the mean value of the prior distribution.

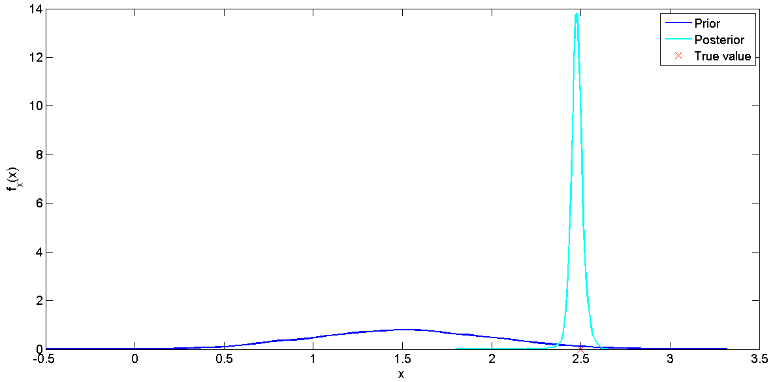

Figure 4.12: Posterior *pdf* obtained applying NL-MMSE and considering the true value located at 2σ.

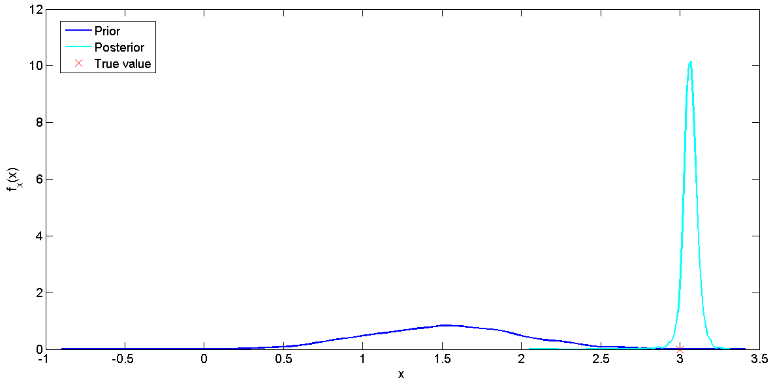

Figure 4.13: Posterior *pdf* obtained applying NL-MMSE and considering the true value located at 3σ.

4.5 Summary

The Bayesian approach to the stochastic inverse problem has been presented. A functional approximation of the input random variables is introduced for quantifying the uncertainty on the model output. This approach has been also shown in practical terms on a toy example. Once that the forward problem is solved, it is possible to exploit the obtained response surface for the identification of the input parameters and the updating of their prior distribution. Three procedures based on general polynomial chaos expansion have been especially considered for solving the problem: the Markov Chain Monte Carlo algorithm, the Polynomial Chaos Expansion based Kalman Filter and a method based on the Minimum Mean Squared Error estimator. Sensitivity analyses have been developed considering simple analytical models characterized by a different degree of non-linearity and varying both the measurement error and the location of the true value. Results confirm that MCMC is a very general and trustworthy method; however it remains time-consuming, even with the functional approximation of the system response, and it is very inefficient in some of the considered cases. The outcomes obtained applying PCE-KF show that a linear update does not perform well in case of non-linearity of the model. Better results are obtained applying the NL-MMSE. This approach succeeds in identifying the input parameters in all the considered cases. However, the fact that succeeds also when the true value is very far from the mean of the prior distribution can be a double-edged sword, since it might be the case that the prior is wrong, and results are not reasonable. All the methods should be thus applied with caution and the results should be carefully evaluated in light of the aspects characterizing the problem that has to be solved. If MCMC does not perform efficiently, it could be a sign that the prior distribution is not correct.

Chapter 5

Evaluation of statistical parameters of concrete strength from secondary experimental test data

The results of material acceptance tests or in-situ tests are a valuable source of information for the reliability assessment of existing structures. Huge secondary databases of test results are usually available, coming from different sources, but individual results are often not associated with a given population. For this reason, their statistical analysis is a complicated process. In order to solve this problem, this research presents a methodology that allows identifying homogeneous populations (or material classes), together with their statistical parameters, when mixed in arbitrary and unknown percentages in a secondary database. The methodology is based on the cluster analysis of data applying the Expectation-Maximization algorithm, which allows to figure out individual classes and their characterizing statistical parameters by fitting a Gaussian mixture model. The proposed methodology has been applied to a relevant case study, investigating the cubic concrete strength of the Italian production during the 1960s, also using different approaches. The study demonstrates that approximately six concrete classes can be identified, characterized by a standard deviation of about 4.0-4.5 MPa, practically independent on the strength. As the results obtained with different approaches agree satisfactorily, it can be concluded that, if enough experimental data are available, the proposed procedure is not only suitable for the intended applications, but it is also "robust" enough.

Contents

5.1 Introduction

In the reliability assessment of existing structures, the estimate of the mechanical properties of the building materials and the evaluation of their most relevant statistical parameters is a crucial issue of the analysis. In some cases, useful information about this topic can be derived, when available, from standard acceptance test results or by in situ investigation. Standard acceptance tests are devoted to assess the mechanical properties of the material and to determine whether a production lot of the material itself is fulfilling the design requirements or not. This important quality control technique has become more and more common in the engineering practice starting from the second half of the 19*th* century, with different emphasis depending on the building material. Usually, material samples are collected in factories, in the framework of the continuous production control, or at the building site, to assess whether the material can be accepted or not. Moreover, often in situ investigation is frequently needed to supplement or to substitute laboratory tests.
As soon as the process for sampling and testing specimens has been codified in relevant Codes and Standards, test results have been stored in laboratory archives and databases; therefore in most of the cases they can be easily retrieved and consulted, regardless of whether they pertain to laboratory or in situ tests.
In a very general context, the statistical parameters of the mechanical properties can be obtained by:

1. analyzing sets of semi-destructive in situ test results; or
2. analyzing non-destructive in situ test results; or
3. prior evidence; or
4. suitable combination of the above-mentioned information.

Nevertheless, the practical application of these approaches is extremely complex and frequently subject to strong restrictions. It must be underlined that:

- semi-destructive tests are, in most cases, incompatible with the statics and the needs of safeguarding the structure or of preserving its cultural value and that, even in those cases when semi-destructive tests can be really carried out, their number is normally so limited that appropriate statistical interpretation is very difficult;

- non-destructive tests are broadly correlated with the actual resistance: their use can be useful to support the identification of reference values of a mechanical property and to assess the homogeneity of the material, but commonly they give no direct information about statistical parameters;
- prior evidence can be derived directly from specific acceptance tests performed on the structure, provided that enough sound and reliable experimental results are available, but this is a very unusual case; or by suitable elaboration of large amounts of secondary data, derived from experimental test results obtained elsewhere. This approach recalls the empirical Bayes method described in Chapter 3, according to which a prior distribution is defined using past realization of the random variable;
- the best way to estimate statistical parameters of material properties is to combine prior evidence, when available, with semi-destructive or non-destructive investigations; by using spot-check results essentially to support the identification of material properties based on prior evidence or, when possible, as a basis for application of Bayesian updating.

The above considerations suggest that only approach (3), alone or in combination, is workable in practice, but it requires the definition of a proper methodology for statistical analysis and interpretation of available secondary data, namely data already collected, even coming from different sources and obtained in different ways or for dissimilar purposes; in effect, as it has been already underlined in Chapter 3, the empirical Bayes approach usually involves some kind of analysis, since data that can be collected is not associated with any realization of the random variable.
Basically, a valuable source of information regarding material strength or more generally mechanical properties is databases that gather sets of results of in situ semi-destructive tests obtained on similar structures or collecting sets of results of standard tests or even assembling semi-destructive and standard test results. Although the material properties of building materials can be generally associated with discrete resistance classes, the statistical analysis of secondary material properties databases is often hindered by the difficulty of identifying in them different resistance classes and then their statistical parameters, since each individual experimental result belonging to the database cannot be referred to a specific resistance class of the material. This observation, which is quite obvious when in situ tests are the exclusive source of information, is valid in a much wider sense, because, even in case standard test results are available, the resistance classes are often not correctly declared or not declared at all;

for instance, use of downgraded material to meet the requirements of a lower resistance class is an emblematic example of incorrect declaration. On the contrary, taking into account that frequently the material properties depend on the composition of the material itself and on the workmanship, and that these aspects are nearly constant in a homogenous geographical area, it seems reasonable to group the data on the base of regional criteria.

The evaluation of the statistical parameters of the compressive strength class of the concrete is a key issue in reliability assessment of existing reinforced concrete structures. Since the execution of standard acceptance or quality control tests on cubic or cylindrical specimens is a common practice in building reinforced concrete structures, huge amounts of test results are available, often further supplemented by test results on cylindrical specimens obtained by in situ core sampling.

The statistical analysis of these secondary data could shed a light on the statistics of the mechanical properties of the existing concrete structures.

An attempt to perform some kind of statistical elaboration of massive test results can be found in literature [166], but it was unsuccessful, as proven by the unrealistically high values of the COV generally resulting from the analysis.

In performing the analysis it must be duly taken into account that preliminary manipulations of the recorded data, like ”a priori” assignment of some specimens to a given class on the base of information recorded on the test report or on the base of engineering judgments, should be avoided as this kind of information is often unreliable or incomplete and anyhow extremely subjective.

It must be underlined that the existence of discrete material classes in general, and of discrete concrete classes in particular, can be interpreted as an intrinsic feature of the material making process, independently on its codification; the unique difference is that in standardized productions notional resistance classes are defined. In effect, also in a first stage and in absence of any standardization, building materials are produced fulfilling some mechanical requirements, suitably adapting the production and the mix design on the base of the past experience and of the raw materials locally available, according to the know-how developed in limited context. Since resistance classes, even not standardized, are nothing more than sets of required mechanical properties, their existence is a natural consequence of the production.

Aim of the present research is to illustrate an ”objective” method for the identification of concrete classes and the evaluation of their most relevant statistical parameters: namely, mean value, standard deviation and COV, starting from a database where secondary data, consisting of compressive strength results obtained on specimens belonging to different concrete classes, are roughly collected.

The basic idea of the method is to partition the mechanical test results by means of a cluster analysis based on Gaussian mixture models (GMMs), in such a way that homogenous statistical populations can be identified. The method, which is very general and applicable to any building material, is also improved with supplementary criteria of acceptance, based on previous knowledge.

The mixture model, in which each cluster is defined by an appropriate *pdf* of the strength, provides results which can be directly used for the reliability assessment of existing buildings. Once those clusters, and thus material classes, have been identified, it is also possible to determine the uncertainty affecting the statistical parameters. The proposed procedure, that is explained in general terms in the following, is also applied to solve a relevant case study, concerning the identification of homogeneous resistance classes in the Italian concrete production during the 1960s, and the evaluation of their relevant statistical parameters.

It must be stressed that allowing to consider large amounts of experimental data, the method offers the opportunity to evaluate the COV associated to each concrete class, which cannot be reliably obtained with the usual approaches, commonly based on a limited number of experimental results.

5.2 General methodology

The proposed procedure, which is very general, is based on the GMM, as illustrated below. The procedure can be applied whenever uncertainties regarding material classes or other relevant mechanical properties and their statistical parameters exist, provided that enough data can be collected. A fundamental assumption of the present proposal is that the *pdf* of the relevant property under examination can be approximated by a Normal distribution, although it could be easily extended and generalized.

5.2.1 Documentary search and literature review

First of all, a documentary search should be carried out, with the objective of improving the knowledge about the range of variation of the material property to be identified as well as of its relevant statistical parameters. In this phase, studies should encompass not only guidelines but also scientific and historical documentation issued at the time when the structure was built, following their evolution over the years, up to the more recent investigations. Other sources of important information might be past Codes and Standards dealing with building materials, as well as treatises, manuals and technical literature summarizing in written form the existing empirical knowledge about the building practice. If the material of interest was produced and only later transported on the building site, it is also possible, sometimes, to make reference to guidelines released by the producer company. This documentary search is also aimed at understanding the physical reasons behind the existence and number of material classes, typical values and expected general trends of their statistical parameters; in fact, it will serve as a basis for checking the soundness and the appropriateness of assumed hypotheses, as well as for supporting the implementation of mathematical algorithms and the evaluation of the obtained results, also resorting to sensitivity studies.

5.2.2 Collection of data

As already stated, the database to be analysed should mainly consist of test results obtained on standardized specimens adequately representative of the building material of the structure to be assessed.
To be adequately representative, tested specimens should be consistent with the investigated structural material in terms of raw materials and workmanship; therefore they should refer to structures coeval to the considered one and belonging to the same geographical region or even to the structure itself.
Once the above-mentioned conditions are satisfied, every single datum can be the result of both standard acceptance material tests carried out on available samples collected at building sites and in situ sampling and testing on similar structures.
Cleary, being filed in laboratory archives or electronic databases, groups of results of standardized tests can be easily collected and combined even if obtained by different laboratories, in such a way that large sets of data can be

elaborated. At the same time, the outcomes of statistical analysis carried out on only smaller parts of data or sub-database can be easily generalised and supplemented, also aiming to determine how relevant parameters vary from year to year. One relevant issue to be tackled in the analysis concerns the minimum amount of data needed for statistical elaboration. This minimum is influenced by several factors and cannot be 'a priori' determined, but it should cover a suitable time interval, typically some months or one year, such as the assumption that each resistance class represents a homogeneous population, whose statistical parameters are independent on time, is justified.
To establish if available data are sufficient or further data are needed, a sensitivity analysis could be performed, considering gradually increasing time intervals, until consistent outcomes are achieved.
However, it should be underlined that:

- insufficient number of data could imply that some clusters, and then some resistance classes, are disregarded, and/or that the statistical parameters of some class are not correctly assessed;
- conversely, too many data could complicate the identification of the various clusters.

For these reasons, the amount of data should be properly balanced to effectively pursue the aim of the investigation. In effect, a general strategy to determine the required amount of data does not exist, and it has to be defined mainly according to researcher's experience and engineering judgments, critically discussing relevant outcomes of the investigations, including appropriate sensitivity analysis.
If the quantity of data is big enough, comparing the results obtained on different subsets it would be possible to check the validity of the outcomes, as well as to assess time trends, provided that subsets refer to different periods.

5.2.3 Cluster analysis based on Gaussian mixture models

Let $\mathbf{y}_1, \ldots, \mathbf{y}_k$ the realized values of k independent and identically distributed random vectors $(\mathbf{Y}_1, ..., \mathbf{Y}_k)$. A finite mixture model (MM) with k components is the distribution $F(\mathbf{y})$ having the following density [103]:

$$f(\mathbf{y}) = \sum_{i=1}^{K} \pi_i f_i(\mathbf{y}), \tag{5.1}$$

where $f_i(y)$ are the component densities of the mixture and π_i are mixing proportions or weights:

$$0 < \pi_i < 1, i = 1, ..., k, \tag{5.2}$$

$$\sum_{i=1}^{K} \pi_i = 1. \tag{5.3}$$

MMs can be used for twofold purposes: modelling situations in which a single parametric family is unable to provide a satisfactory model and/or k distinct groups are known a priori to exist in some physical sense.
Taking into account that, as already said, materials can be classified according to discrete resistance classes, for the aim of the present work the mixed model is used according to the second purpose, for providing model-based clustering.

5.2.4 Fitting Gaussian mixture models

An MM can be easily fit to a group of data belonging to k different populations normally distributed if the population to which each datum belongs or, equivalently, the statistical parameters of the *pdf* of each population are known. Since these details are usually unknown, an iterative procedure which maximizes the Log-likelihood of the data, called Expectation-Maximization (EM) algorithm, could be applied.
The relevant steps of the algorithm can be summarized as follows:

- k Gaussian distributions are randomly chosen, being k a suitable number;
- the likelihood of each data x_i, corresponding to the probability for x_i to be a sample from a_j, is computed considering each distribution a_j, $j = 1, \ldots, k$:

$$P(x_i|a_j) = \frac{1}{\sqrt{2\pi\sigma_{a_j}^2}} \exp\left[\frac{-(x_i - \mu_{a_j})^2}{2\sigma_{a_j}^2}\right]; \tag{5.4}$$

- the posterior distributions given each data point is computed, being $a_{ij} = P(a_j|x_i)$;
- the statistical parameters of each distribution are re-estimated, according to:

$$\mu_{a_j} = \frac{a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n}{a_{j1} + a_{j2} + ... + a_{jn}}, \tag{5.5}$$

$$\sigma^2_{a_j} = \frac{a_{j1}(x_1 - \mu_{a_j})^2 + a_{j2}(x_2 - \mu_{a_j})^2 + ... + a_{jn}(x_n - \mu_{a_j})^2}{a_{j1} + a_{j2} + ... + a_{jn}}; \tag{5.6}$$

- repeat the previous steps until convergence is reached.

In order to obtain a sound estimation of the statistical parameters of each cluster, the initial values of the EM algorithm should be suitably chosen. In effect, it has been shown that different starting strategies and stopping rules lead to quite different results [144, 145]. In fact, inappropriate choice of the starting values of the algorithm implies twofold drawbacks: the slowness of the convergence usually affecting the procedure is aggravated on one hand, and, if the likelihood function is unbounded on the boundaries of the parameter's space and the initial values are chosen on the boundary, the sequence of estimates may diverge, on the other hand. A common situation is that the likelihood has multiple roots corresponding to local maxima: in this case, the algorithm should be applied to a wide choice of initial values in any search of local maxima. A possible option is to apply the EM algorithm for a number of random starts.
For a mixture model characterized by k components with mean μ_i, it is possible to randomly generate the $\mu_i^{(0)}$ as:

$$\mu_1^{(0)}, ..., \mu_k^{(0)} \approx N(\bar{y}, V), \tag{5.7}$$

where $\bar{y}$ and V are respectively the mean and variance of the sample.
The algorithm fits an MM assuming that the number of distributions k is known. However, this is not always the case; a method to identify the number of components k is discussed in the following paragraph.

5.2.5 Information criteria

If no prior information about the number of clusters is available, and the MM is multi-modal, k could be assumed, as a first attempt, equal to the number

of modes.
But, when each individual component is not sufficiently distant from the others, the MM could be characterized by a number of modes lower than k and the data set could look even unimodal. Furthermore, when the MM is used for providing model-based clustering, the choice of the number of distributions k should be made following the Occam's Razor principle, which states that if multiple models fit equally well a set of data, the simplest one should be chosen. In practice, this proposition can be encoded using log-likelihood criteria, also called information criteria.
The most popular criteria for taking a decision regarding the number of components in the MM are the Akaike Information Criterion (AIC):

$$AIC = -2 \ln \hat{L} + 2p, \tag{5.8}$$

and the Bayesian Information Criterion (BIC):

$$BIC = -2 \ln \hat{L} + p \ln i, \tag{5.9}$$

where the first term, comprising the maximized value of the likelihood function $\hat{L}$, represents the lack of harmonization, while the second term, comprising the number of parameters p, is the penalty term depending on the complexity of the model; in case of BIC the second term also comprises the number of data i.
Despite similarities in their expression, AIC and BIC have a deeply different meaning and depending on the context in which they are applied, one should be preferred to the other. As clarified by [21], *"there are two cultures in the use of statistical modelling to reach conclusions about data. One assumes the data are generated by a given stochastic data model. The other uses algorithmic models and treats the data mechanism as unknow"*. The model that best fits the data must be interpreted differently in the two above mentioned worlds [3]. In the first case, as sample size increases, it is expected to find the correct model; it is thus a matter of confirmation or falsification of the considered models. In the second case, the true model cannot be found; it is thus a matter of selecting the model that maximizes predictive accuracy. Usually, since AIC has the property of efficiency, it outperforms BIC in the latter situation; on the other side, since BIC has the property of consistency, it outperforms AIC in the former circumstance. Furthermore, the above-mentioned criteria can be justified on theoretical bases, arguing that the model with the minimum AIC value should be asymptotically the closest to the true model, according

to Kullback-Leibler distance [151], while the model with the smaller BIC value should be the one with the highest posterior probability.
Finally, several types of research have pointed out that AIC tends to overestimate the number of components; in fact, for $i > 8$ and $ln(i) > 2$, BIC penalizes complex models more heavily than the AIC, so reducing the risk that too many components are fitted.
In the light of the above remarks, the choice of the model selection tool has to be made considering the peculiarities of the situation, especially considering the process generating data and the size of the sample [24]. Assuming that BIC_{min} and AIC_{min} are the minimum values of BIC and AIC, and that BIC_i and AIC_i are the values related to the i^{th} model, the models have been ranked according to the following quantities:

$$\Delta_{BIC,i} = BIC_i - BIC_{min}, \tag{5.10}$$

$$\Delta_{AIC,i} = AIC_i - AIC_{min}. \tag{5.11}$$

As stated in [129] and [25], for $\Delta_{AIC,i} \leq 2$ the evidence of the i^{th} model compared with the model with the minimum value of the information criterion is weak and thus it cannot be rejected. Assuming that further data cannot be collected, it is possible to calculate a weighted average of all the models that cannot be rejected, being the weightw_i computed according to the values of $\Delta_{AIC,i}$ by means of:

$$w_i = \exp\left(\frac{AIC_{min} - AIC_i}{2}\right), \tag{5.12}$$

that basically represents the relative likelihood of the model i.

5.2.6 Identification of material classes

Once the cluster analysis based on GMM has been completed, it would be possible to identify material classes and the related statistical parameters.
In some lucky cases, it will be possible to directly define material classes from the mean value and the standard deviation of each identified clusters. Actually, results obtained so far are generally not so evident, and further studies are required, but this aspect is not particularly relevant, being the standard deviation and the COV the main parameters sought.
In any case, clusters consisting of less than 100 data points and clusters whose

statistical parameters cannot be assessed with precision, or even characterized by unrealistic values of the COV, should be discarded.
Moreover, the presence of some outliers, resulting from very low or very high values of the investigated property, could strongly influence the identification of the extreme clusters. This eventuality is clearly emphasized when extreme clusters are characterized by COVs much higher than expected; in this case too, extreme clusters should be discarded.
Assuming that parameters that mainly identify a material class are the mean value and the COV of the fitted Gaussian distribution, a second cluster analysis could be performed considering the statistical parameters obtained by fitting the GMM and implementing a k-mean algorithm. In this way, relevant material classes could be better identified, although supplementary investigation might be needed to assess the best k value.
It must be underlined that the proposed method is a blind procedure not requiring particular assumptions; consequently, the identification of material classes is markedly objective.

5.2.7 Determination of the uncertainty affecting the statistical parameters

The statistical parameters characterizing each material property are random variables affected by aleatoric uncertainty. The inherent randomness can be due to several factors, but a particularly significant and leading aspect might be represented by quasi-periodic variations over time affecting the quality of the material production process, and consequently the statistical properties of the batch of samples tested in laboratory.
To check how the main parameters of each material class vary from year to year, databases covering a sufficiently protracted period of time can be subdivided into subsets covering a suitable number of time intervals, each one satisfying a minimum amount of data required. Then the results obtained by fitting the MM of each data subset can be inferred as realizations of the statistical parameters characterizing each investigated mechanical property, and appropriate elaboration of these data should allow assessing the inherent aleatoric uncertainty of the mechanical property itself as suggested by the empirical Bayes approach. Adopting a value for k matching the previous analysis, a new cluster analysis, based on a GMM characterized by k components, could be performed considering the above-mentioned database in order to obtain

additional information. Finally, critical comparison of the statistical parameters of the fitted *pdf*s with those previously obtained analyzing separately each data subset should provide arguments to validate the results or to ask for additional investigations.

5.3 Definition of the statistical parameters of concrete resistance classes in the 1960s

The application of the proposed methodology is illustrated in detail analyzing a relevant case study, devoted to assessing the statistical properties of the concrete produced in Italy during the 1960s. The case study is very important in view of reliability analysis of existing Italian reinforced concrete structures and for the planning of strengthening interventions, because a significant quota of them, approximately 20%, were built in that decade and a relevant part needs nowadays to be refurbished.
At that time, the concrete was mostly prepared by in situ mixing, often following some empiric volume metered reference recipes, while the use of ready-mix concrete, weighted metered, was limited to the most relevant structures.
In addition to pointing out again that the mechanical properties and the mix design declared in the test report are often missing or incorrect, it must be stressed that mechanical properties of site-mix concrete were hardly predetermined, because the use of reference recipes could not take account that concrete properties depend not only on the recipe itself but also on the type of cement and on the aggregates used for the preparation, as well as on the compaction degree. These remarks confirm that the procedure should disregard this kind of information and that resistance classes should be determined simply seeking for individual clusters in the resistance database, in such a way that samples belonging to a given class are recognized on the base of their properties and not on preconceptions.

5.3.1 Documentary search and literature review

5.3.1.1 The coefficient of variation of concrete strength

As already stated, additional criteria for the acceptance of the outcomes of the proposed procedure could be introduced, in particular, based on the value

of the COV. Relevant literature has been thus preliminary examined to assess some reference values for COV.
The concrete strength, whether it is produced on site or ready mix, is a random variable, whose statistical parameters can vary over time, even in a single structure. According to [161] and [61], several factors can affect the variation of concrete properties, like the size of the job and the duration of the contract, the supervision, workmanship and plant used, the making, curing and testing of the specimens, the variation in successive batches, and the variation in the constituent materials. Above all, and especially on site, a relevant source of the variability is due to the fluctuations of water to cement ratio (w/c), caused by the continuous adjustment of the quantity of water added at the mixer in attempting to maintain a good level of workability and the variation in moisture of the aggregates, as well as the climatic conditions during the preparation and pouring of concrete.
The w/c is basically considered the most influencing single factor in the strength of fully compacted concrete [119]. Several empirical relationships between concrete strength and w/c have been proposed; for example, some references can be found in [119] and [138], where no specific information is given about the influence of w/c on COV. Moreover, [61] proposes to refer to the w/c curve applied to the minimum strength rather than to the mean value, applying the control function to the w/c rather than the strength. But the concrete class cannot be automatically derived from the w/c, even when it is perfectly known; in fact, the mix design aiming to achieve a given concrete class operates on all relevant factors and not only on the w/c. In this respect, it should also be recalled that several types of cement exist and that even the range of variation of the standard compressive strength of cement can be large; for example, for cements belonging to both classes 32.5 and 42.5 [53] and [46] prescribe a range of variation of 20 MPa. In several works [114, 89, 115, 52] the distribution of the strength of test specimens is assumed to be Gaussian and thus can be described by the mean value and the standard deviation. For practical purposes, the assumption of normal distribution is acceptable, although examples of skewness have been reported by [113], for low strength concrete, and by [33] and also in [2] for high strength concrete. Usually, the assumption of normal distribution errs on the safe side with respect to the number of test results expected to fall below the specified value of strength. [61] noticed a slight skewness of the distribution for low strength and high strength concretes and longer tail for medium strength concretes, but departure from normality does not have great effect, unless stricter minima are

specified for low or high strength concretes, because in these cases, adoption of a normal distribution may result in a bad fit. A lognormal distribution might be more appropriate here [82], but, as known, when COV is below 20%, like for concrete strength, the characteristic strengths, defined as 5% lower fractile, determined considering normal distribution or log-normal distribution are very close together.

Concerning COV, it is still unclear if concrete classes with increasing strength are characterized by constant standard deviation, constant COV, or none of the two. Moreover, this issue is deeply related to acceptance criteria of experimental results and in particular on the relationship between the mean and the minimum experimental strength adopted in the acceptance criteria. Of course, if COV is assumed independent of the concrete strength, the ratio between the mean and the minimum value is constant; if the standard deviation is assumed independent on the concrete strength, the difference between the mean and the minimum value is constant. Studies carried out in [161] on ready-mix and site-mix concrete productions concluded that COV is independent of the strength, as confirmed by some laboratory studies; while most recent studies related to site conditions demonstrated strength independence of the standard deviation [119], even if the question is still discussed. In a past version of the Swiss Standards, characteristic values were defined considering a COV of about 20%, independent on the strength, while in more recent versions [147], a constant value of the standard deviation is assumed. Some authors came to the conclusion that the standard deviation is strength independent at levels exceeding a certain strength. According to [116] and [1], the standard deviation increases linearly until resistance of about 20 MPa, then it remains constant irrespective of the strength; for [61, 137, 161, 120] the relationship between the standard deviation and strength can best be represented by a smooth curve through the origin tending to become a horizontal straight line for values of mean strength bigger than 28 MPa.

In [149] it is concluded that both standard deviation and COV depend on strength, being the standard deviation less sensitive to strength variation than COV; the overall average standard deviation found by Soroka is around 6.0 MPa, in agreement with the data published in [61], where values between 5.9 and 6.2 MPa are derived for fair control conditions, and in [137], where values between 5.0 and 6.0 MPa are deduced. However, in some cases relationships are proposed based on different conclusions, assuming the standard deviation depending on the strength or even on actual site conditions. Following [168], the COV is independent of the concrete strength but dependent on the degree of control exerted over the production process. Moreover, as the control conditions were not exactly the same in all sites covered by this study, it might be argued that evaluation of the same data by a method more sensitive to the actual level of quality control, would have led to a different conclusion. In general, the findings of most of the above-mentioned studies lead to the conclusion that standard deviation is independent on the concrete class, as it is also implicitly assumed in [56, 65, 121] for quality control purposes. Moreover, the outcomes of the case study illustrated in the following corroborate this conclusion.

5.3.1.2 The production of concrete in Italy during the 1960s

The case study refers to the Italian concrete production in the 1960s; therefore it would be very helpful to summarize the main features of the concrete industry in Italy at that time. As recalled in [125], after the Second World War the Italian cement industry developed at an appreciable rate, adequate to the rapid economic growth of the Country during the so-called "Economic boom". In effect, during the 1960s, Italy became the first in Europe for concrete production and among the first worldwide for concrete technology, also due to the huge number of reinforced concrete structures built in that period.
The control of the total amount of water, with the aim of maintaining w/c constant in order to reach the prescribed quality, was sensibly improved in concrete batching plants, where mixers were equipped with special devices for controlling the quantity of water fed called 'concrete hygrometers' [36]. Nonetheless, the old system of mix-water control was still widely applied, consisting of a suitable regulation of the water fed into the mixer, on the basis of the measured moisture content of aggregates, but resulting in a wider scattering of the concrete strength. On the other hand, in most cases, concrete was still directly produced on the building site on hand-made bases. In this condition, to obtain concrete with the required properties, the concrete producer, who was often a construction worker, mixed cement, water, aggregates according to empirical recipes, based on his experience and on the raw materials locally available, so determining unavoidably large scattering of the hardened concrete properties. The Italian Code governing all the activities connected to the building industry in force at that time was still the one specified in [48]. Concerning the acceptance tests on concrete, the sampling of four standard cubes was required every 500 cubic meters of concrete casting. The standard

cubes, having a side of 160 mm or 200 mm depending on the maximum size of aggregates (smaller or greater than 30 mm respectively), were tested after 28 days of curing, ordering the results in decreasing order, from σ_1 to σ_4, excluding the minor outcome σ_4 and determining the average cubic resistance f_c:

$$f_c = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}. \tag{5.13}$$

The test was considered successful if both the following conditions were verified:

$$f_c \geq f_c^\star, \tag{5.14}$$

$$f_c \geq 12 \quad MPa, \tag{5.15}$$

where $f_c^\star$ was the design strength.
For each test, the following data were recorded: date of sampling; date of testing; cube dimensions; effective area of the cross-section of the specimen; ultimate load and compressive strength. Other data were possibly recorded such as type, origin and amount of cement and aggregates; building site; structural element from which cubes were sampled; the applicant of the test and so on. It must be highlighted that on the form to request tests on cubes there was no information regarding the strength adopted in the design or the concrete class, even because in Italy formal reference to design classes and characteristic values was introduced successively, in 1972, by the decree [49], when the concrete was a standardized and more industrial material, produced in concrete plants and later delivered on site.
According to the literature review summarized above, it has been possible to speculate over the number of concrete classes and their statistical parameters. The most influential Italian reference at that time was probably the already cited [138], because it was a widely known and implemented practical manual for both engineers and concrete producers. In that book, at least 7 classes were indicated, just as many w/c, according to which it was possible to obtain a given resistance, ranging from 17 to 50 MPa. Concretes characterized by a very poor quality, to be used in non-structural elements, and barely satisfying the condition expressed in Eq. 5.15, could envisage a very low class; while high strength concrete, typically employed in pre-stressed concrete structures, could ask for the definition of further class.

5.3.2 Collection of data

In a first stage, it was analyzed only the year 1965, for which 3725 results of single compressive tests were available. The preliminary cluster analysis performed on them, whose results have been already published [108], shows that, as expected, the mixture model is very promising, so encouraging further investigation. The analysis was then considerably widened, taking also into account test results available for the years 1961 (3379), 1963 (4816), 1967 (6302) and 1969 (6221), in order to have representative data samples covering the whole decade. The huge number of test results (18222 in total) embraces all relevant structural typologies, such as residential and public buildings, industrial structures, roads, bridges, hydraulic structures and foundations.

5.3.3 Preliminary analysis of the histograms

The compressive cubic strength results are plotted in the relative frequency histograms illustrated in Fig. 5.1, 5.2, 5.3, 5.4, 5.5, referring to the years 1961, 1963, 1965, 1967 and 1969, respectively, while all available data are grouped in the histogram in Fig. 5.6. Observing the histograms, it is possible to draw some preliminary conclusions, in particular about the existence and the number of classes to be considered in the elaboration:

- most of the histograms, in particular the one referred to 1961 in Fig. 5.1, show a longer tail on the right, in which sub-models are clearly emerging;
- the histogram related to 1963 in Fig. 5.2 appears almost symmetrical, but a more deep look reveals at least 7 clusters;
- the histogram related to 1965 in Fig. 5.3 allows a very easy identification of sub-models, as 8 or 9 clusters of data can be recognized by a visual check;
- the histograms related to 1967 in Fig. 5.4 and 1969 in Fig. 5.5 are much more smooth, but in spite of this they clearly reveal two modes in the range from $20-40\ MPa$;
- besides it implies the loss of the information about variations over time, the global histogram in Fig. 5.6 shows less marked modes than individual histograms, therefore it has been used only in a subsequent

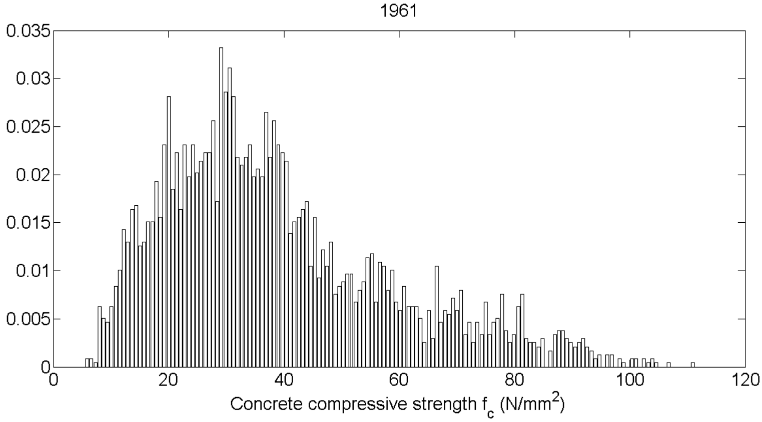

Figure 5.1: Histogram of results about compressive cubic strength of concrete - year 1961.

phase to validate or to accept the results derived analyzing the individual histograms.

A number of 7-10 concrete classes, each one characterized by its own *pdf*, could be expected accordingly. The modes and the mean values of two adjacent distributions should be equally spaced, while the distance between their characteristic values, or more generally between their x-fractiles, depends on their standard deviations. Obviously, uniformly spaced x-fractiles indicate that the standard deviation is independent of the concrete class.

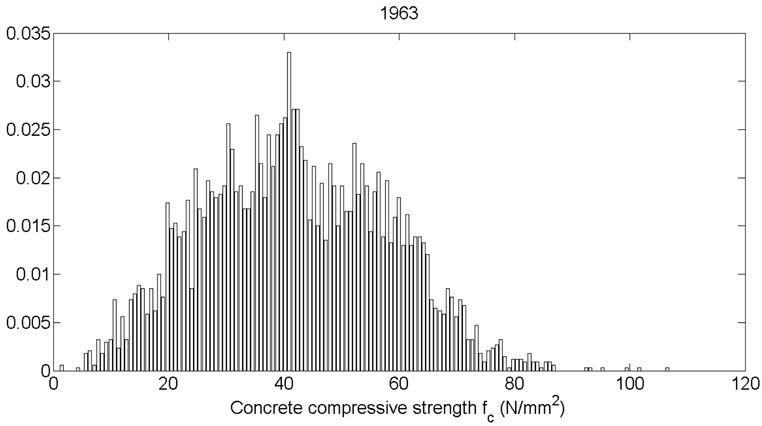

Figure 5.2: Histogram of results about compressive cubic strength of concrete - year 1963.

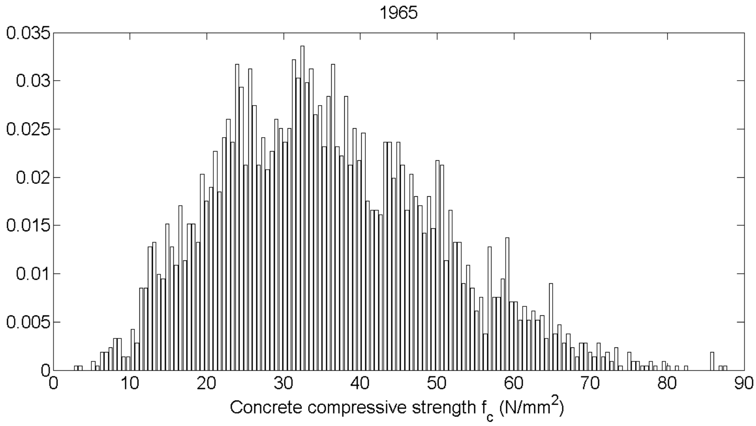

Figure 5.3: Histogram of results about compressive cubic strength of concrete - year 1965.

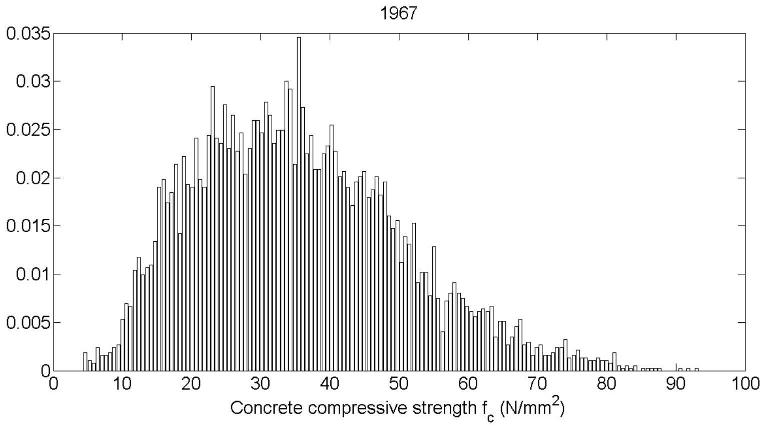

Figure 5.4: Histogram of results about compressive cubic strength of concrete - year 1967.

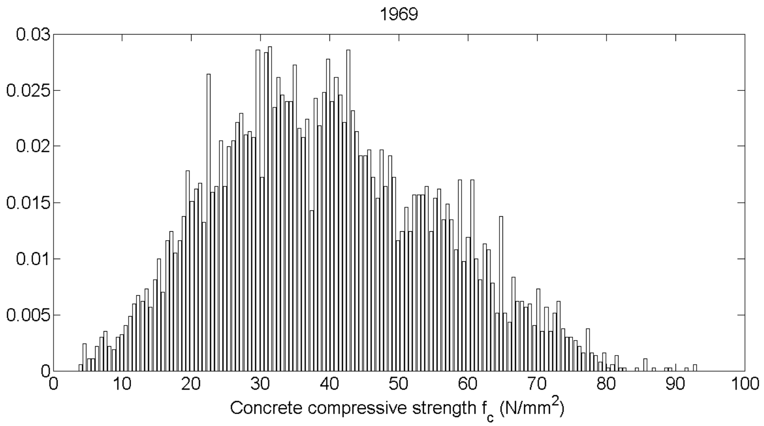

Figure 5.5: Histogram of results about compressive cubic strength of concrete - year 1969.

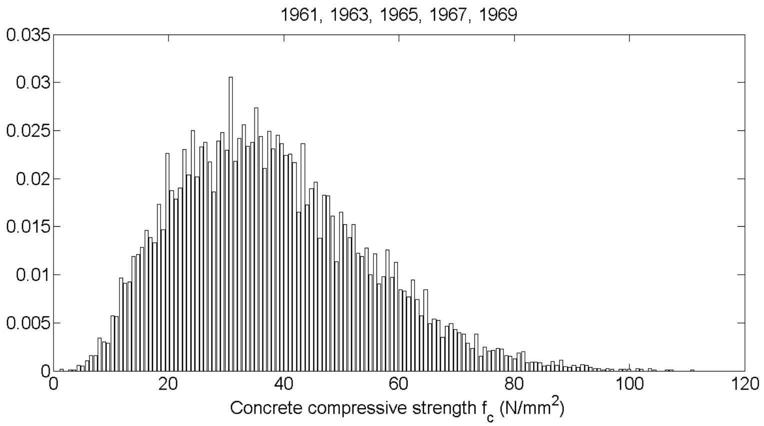

Figure 5.6: Global histogram of the compressive cubic strength of concrete results for years 1961, 1963, 1965, 1967 and 1969.

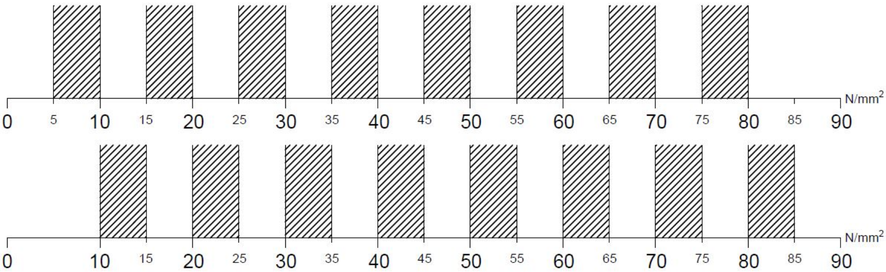

Figure 5.7: Intervals considered for sampling starting values ($k = 8$).

5.3.4 Cluster analysis based on Gaussian mixture models

The available data have been analyzed with the EM algorithm considering a number of components, k, variable from 7 to 10. For each assumed value of k, the algorithm has been run considering different initial values. More precisely, for each i^{th} component, $i = 1, \ldots, k$ the starting point $\mu_i^{(0)}$ has been calculated performing a MC simulation in plausible ranges: 1000 samples $\mu_i^{(0)}$ were drawn from a uniform distribution $U(x_{i,min}, x_{i,max})$, hypothesizing that:

$$(x_{i,min}, x_{i,max}) = 5 \quad MPa, \tag{5.16}$$

$$(x_{i+1,min}, x_{i,max}) = 5 \quad MPa, \tag{5.17}$$

and setting the left bound of the first interval, $x_{1,min}$, the values 2.5 MPa, 5.0 MPa or 10.0 MPa, as appropriate.
For $k = 7$, also the following case has been considered:

$$(x_{i,min}, x_{i,max}) = 10 \quad MPa. \tag{5.18}$$

In effect, wider intervals were here necessary to cover the total range of the whole sample.
The intervals which have been considered in cases $k = 8$ and $k = 9$ for sampling starting values are shown in Fig. 5.7 and 5.8, respectively. Eq. 5.16 and 5.17 lead to the following conditions:

$$5 \quad MPa < \mu_{i+1}^{(0)} - \mu_i^{(0)} < 15 \quad MPa, \tag{5.19}$$

reasonably implying that the distance between the mean values of two adjacent clusters varies in the range $5 - 15$ MPa. Running the EM algorithm according

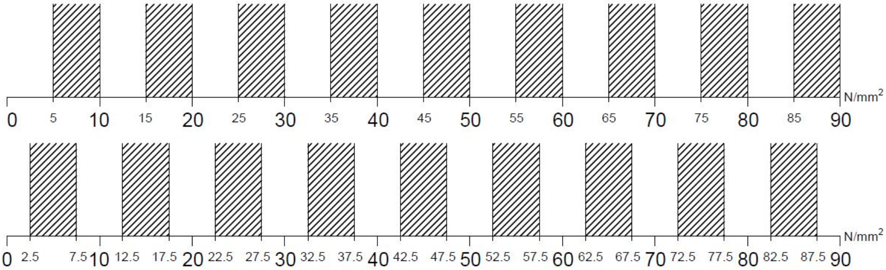

Figure 5.8: Intervals considered for sampling starting values ($k = 9$).

to the above-described procedure, it emerged that, after a few runs, for $k = 7$ and $k = 10$, the covariance matrix became ill-conditioned, because at least one component was characterized by not enough observations with significant weights. On the contrary, for $k = 8$ and $k = 9$ the algorithm ran correctly, providing reasonable results.
At the end of each run AIC and BIC values were computed, so that the obtained models were ranked according to $\Delta_{AIC,i}$ and $\Delta_{BIC,i}$, for both $k = 8$ and $k = 9$.
Taking into account all the models that could not be rejected, AIC and BIC values were evaluated on the weighted model, averaging over the statistical parameters:

$$\bar{\mu} = \frac{w_1\mu_1 + w_2\mu_2 + ... + w_n\mu_n}{w_1 + w_2 + ... + w_n}, \tag{5.20}$$

$$\bar{\sigma} = \frac{w_1(x_1 - \mu_1)^2 + w_2(x_2 - \mu_2)^2 + ... + w_n(x_n - \mu_n)^2}{w_1 + w_2 + ... + w_n}, \tag{5.21}$$

where the w_i are given by Eq. 5.12. As already explained, when $\Delta_{AIC,i} \leq 2$ the i^{th} model cannot be rejected, therefore for $i = n$ it results $\Delta_{AIC,i} \approx 2$, so that minimum meaningful value $w_n = exp(-1) \approx 0.368$.
In order to identify the number of components, the averaged models were compared in terms of AIC and BIC. It emerged that lower values of BIC were obtained for $k = 8$, while the AIC values resulted very close together, as the difference was less than 2.0. These results are coherent with the above discussion about the criteria, according to which BIC tends to penalize more complex models. Plotting the *pdf*s revealed that, for $20 < f_c < 50\ MPa$, clusters identified considering eight or nine components basically coincided; fitting nine components, an additional cluster was obtained, usually for very high or very low resistance, which was outside the field of interest, as characterized by large scattering and including a limited amount of data. For this reasons,

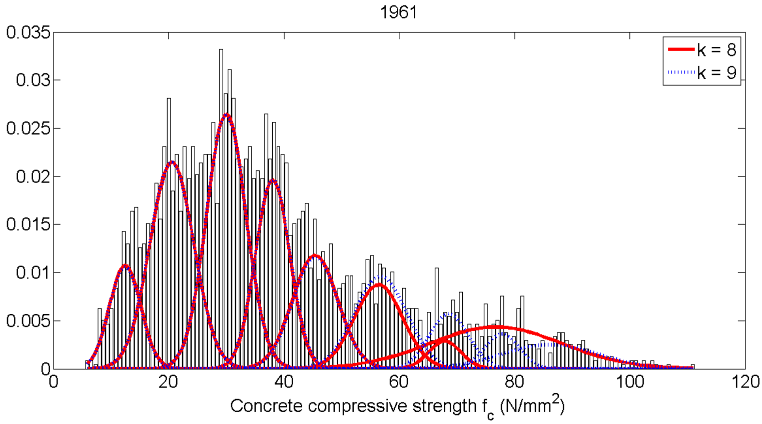

Figure 5.9: Histogram of test results and MMs for $k = 8$ (solid red lines) and $k = 9$ (dotted blue lines) – year 1961.

the final decision was to consider $k = 8$, disregarding the extreme clusters, influenced by outliers, as well as any other cluster comprising less than 100 data.

Fig. 5.9, 5.10, 5.11, 5.12 and 5.13 display the histograms as well as the best MMs obtained considering eight or nine components, for each annual database. In the figures, which are sorted in chronological order, the clusters identified by the eight component MMs are plotted with solid red lines, while the clusters identified by the nine component MMs are plotted with dotted blue lines. The statistical parameters, mean value and COV, of each component of the annual distributions as well as the percentage of data belonging to each cluster, have been summarized in Table 5.1, where values concerning clusters that have been discarded are on grey background. As discussed before, clusters have been discarded when characterized by too low or too high unrealistic values of COV or when characterized by an inadequate number of elements.

One important observation clearly emerges looking at the values of the COVs of the models fitted on yearly data. COVs range in the interval $\approx 0.06 - 0.25$, but when the mean value of the cubic strength is bigger than 20.0 MPa, COV is generally lower than 0.18. High values of COV are typical for low resistance concretes; as soon as the concrete strength increases COV, tends to reduce so that for high strength concrete it results in $0.07 - 0.09$.

On the contrary, standard deviation remains nearly constant and independent of the concrete strength, being around $4.0 - 4.5$ MPa. Besides, it must be underlined that clusters identified in different annual databases are charac-

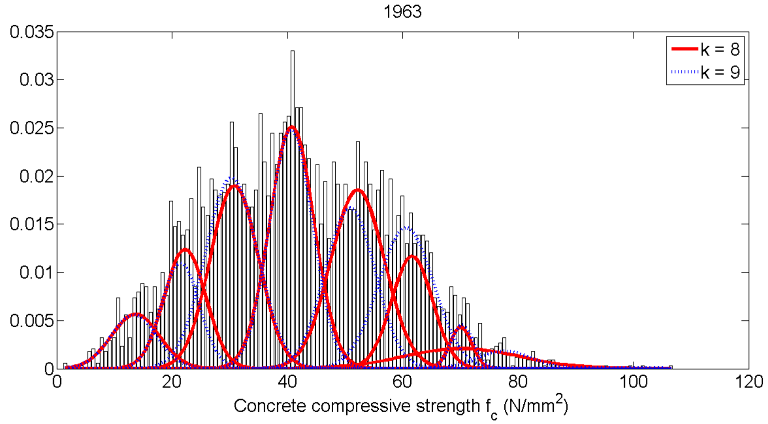

Figure 5.10: Histogram of test results and MMs for $k = 8$ (solid red lines) and $k = 9$ (dotted blue lines) – year 1963.

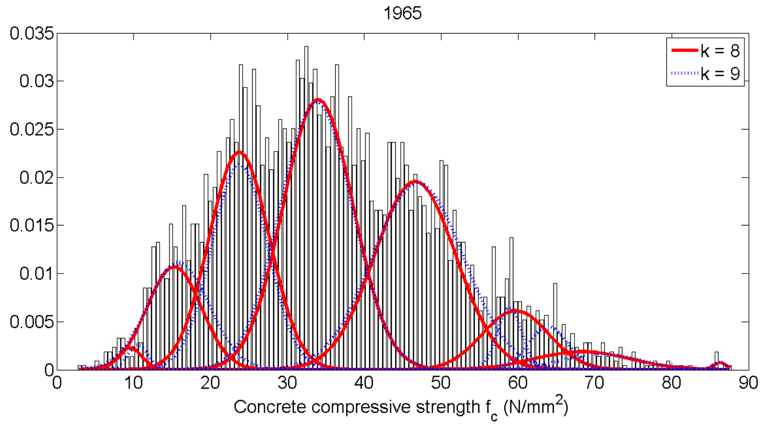

Figure 5.11: Histogram of test results and MMs for $k = 8$ (solid red lines) and $k = 9$ (dotted blue lines) – year 1965.

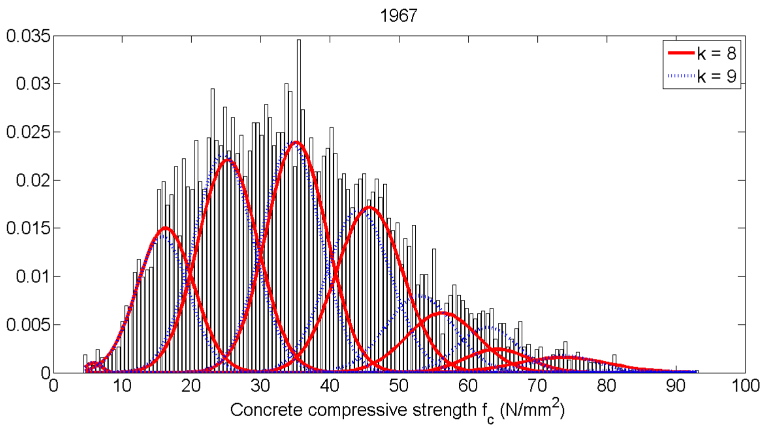

Figure 5.12: Histogram of test results and MMs for $k = 8$ (solid red lines) and $k = 9$ (dotted blue lines) – year 1967.

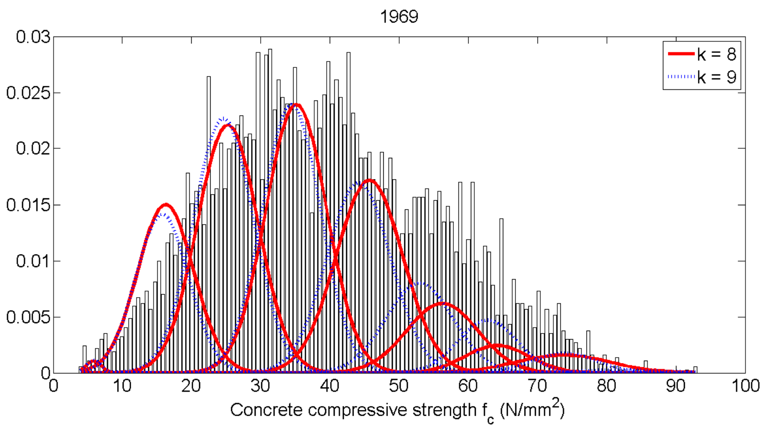

Figure 5.13: Histogram of test results and MMs for $k = 8$ (solid red lines) and $k = 9$ (dotted blue lines) – year 1969.

		Components							
		1	2	3	4	5	6	7	8
1961	μ $[N/mm^2]$	12.5	20.6	30.0	38.1	45.4	56.4	67.9	76.8
	COV	0.214	0.183	0.114	0.077	0.087	0.074	0.041	0.149
	PC	0.072	0.204	0.227	0.145	0.116	0.091	0.020	0.125
1963	μ $[N/mm^2]$	13.7	22.2	30.8	40.8	52.2	61.6	70.1	70.2
	COV	0.303	0.160	0.134	0.094	0.089	0.057	0.026	0.143
	PC	0.059	0.111	0.197	0.242	0.215	0.103	0.020	0.052
1965	μ $[N/mm^2]$	9.5	15.3	23.7	34.0	46.6	59.7	68.4	86.2
	COV	0.160	0.230	0.164	0.134	0.114	0.072	0.090	0.009
	PC	0.009	0.094	0.221	0.320	0.260	0.066	0.029	0.002
1967	μ $[N/mm^2]$	5.8	16.3	25.2	35.1	45.7	56.2	64.2	73.7
	COV	0.179	0.248	0.172	0.125	0.105	0.092	0.071	0.090
	PC	0.003	0.152	0.241	0.263	0.207	0.080	0.028	0.027
1969	μ $[N/mm^2]$	13.1	22.3	31.8	41.8	51.7	60.5	71.3	79.9
	COV	0.330	0.187	0.129	0.095	0.092	0.072	0.061	0.090
	PC	0.057	0.168	0.245	0.220	0.150	0.107	0.048	0.006

Table 5.1: Statistical parameters of the MM components and percentage of data belonging to each cluster for $k = 8$. PC is the probability component, or in other words the probability of the data to belong to that cluster.

terized by comparable statistical properties and that, in particular, clusters characterized by comparable resistances have similar COVs. These results confirm that the proposed procedure is able to identify properly homogenous populations of test results, although the origin of individual data is unknown. Anyhow, looking at the identified clusters, it appears difficult to directly draw some conclusion about the concrete classes. For this reason, a further analysis based on k-means has been carried out, considering the statistical parameters (μ and COV) of the fitted Gaussian distributions.

5.3.5 Identification of material classes through k-means algorithm

As each material class can be represented in terms of mean value and standard deviation, a second cluster analysis was performed, based on the so-called k-means algorithm, in order to identify more precisely concrete classes. The k-means clustering is a method where a group of n observations is partitioned into k clusters in such a way that each observation belongs to the cluster with the nearest mean value. In other words, the mean of each cluster, that here is especially called prototype, should be chosen such that the variance of all the clusters is minimized, thereby seeking for compact groups of observations. Assuming that the space of the prototypes M is identical to the data space $X = \mathbb{R}^m$ and that some suitable metrics d, $d : M \times X \rightarrow \mathbb{R}^+$, like the Euclidean distance, allowing to directly compare clusters and data is defined, the location of each cluster can be easily derived, provided that the number of clusters and the data belonging to each cluster are known.
Since data belonging to each cluster are generally not known, it is necessary to set up a proper method to recognize them.
Let $p_{i|j} \in (0, 1)$ a binary membership matrix that expresses whether the datum $x_j \in D$ belongs to the i^{th} cluster C_i, the k-means clustering starts with some initial guesses of the prototypes; successive updating is performed alternating the adjustment of the membership matrix and the adjustment of the prototypes. The updating can be carried out according to various algorithms. The k-means algorithm defined before optimizes the following objective function J_{kM} during this iterative refinement:

$$J_{kM} = \sum_{i=1}^{k} \sum_{x \in C_i} \left\| x_j - p_i \right\|^2 , \tag{5.22}$$

where p_i is the mean value of data in C_i. Further details regarding other possible algorithms are given in [98].
The application of the k-means algorithm to the case study required to define the number of clusters to be taken into account. By looking at the scatter plot, it emerges clearly that at least two roughly hyper-spherical clusters could be identified for $\mu < 40\ MPa$ and $COV > 0.1$; on the other hand, it is much more difficult to identify groups in the remaining data, but at least three or four clusters could be hypothesized. The algorithm was run again considering six or seven clusters, establishing the initial conditions similarly to the cluster analysis based on GMM described before, namely considering that clusters' prototypes should be located at regular intervals of around 10 MPa. Anyhow, running the algorithm with different initial conditions led to almost the same results, probably as a consequence of the limited number of data. By comparing the results, it turns out that the most reasonable partition is obtained considering seven clusters. Clusters representing groups of data are highlighted in Fig. 5.14 with different colours, together with their centroids. Each class could be reasonably inferred as a concrete class, whose average statistical parameters, corresponding to the coordinates of the centroid of each class, are reported in Table 5.3; since the first cluster has been discarded, parameters referring to it are on grey background. The results, in terms of mean value and standard deviation of concrete resistance of each cluster and in terms of COVs, are in line with those previously determined. The agreement is even more significant considering that, as anticipated, rather than the identification of individual concrete classes, which could be assessed by means of simplest methods, the main scope of the analysis is to ascertain standard deviation and COV and their dependence on the concrete strength.

5.3.6 Determination of the uncertainty affecting the statistical parameters

In principle, starting from the partition derived applying the k-means algorithm, it could also be possible to assess the uncertainty affecting the statistical parameters characterizing individual resistance classes, that could be summarily represented by the COV of each of them. But, in the present case study the number of realizations, namely the number of years considered in the database, is too low for defining a class-related uncertainty. Therefore, an overall uncertainty has been defined, which has been estimated considering all the

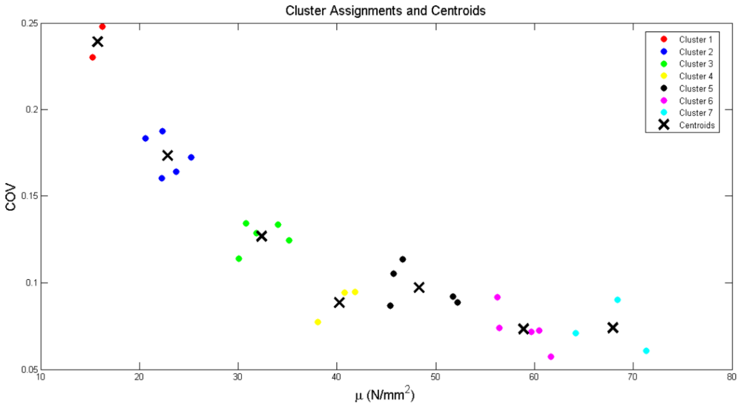

Figure 5.14: Cluster assignments and centroids.

realizations of the statistical parameters in their whole, without referring to any particular class.
In order to assess the uncertainty, for each n^{th} year taken into account, the values of the parameters $x_{i,n}$ of the i^{th} cluster identified by the k-means have been firstly normalized to $x'_{i,n}$ with respect to the prototype p_i of the cluster itself:

$$x'_{i,n} = \frac{x_{i,n}}{p_i} \tag{5.23}$$

and, subsequently, all the normalized $x'_{i,n}$ parameters have been grouped in a unique set $\dot{X}$ including all the years. Once determined the median $\tilde{x}$ of the elements belonging to $\dot{X}$, a further normalization has been carried out with respect to $\tilde{x}$:

$$x''_{i,n} = \frac{x'_{i,n}}{\tilde{x}} \tag{5.24}$$

so that a new set $\ddot{X}$ has been obtained, whose elements $x''_{i,n}$ are characterized by a median equal to 1.0.
The histograms of μ'', σ'', and COV'', of concrete resistance, as derived from the analysis of the case study data and normalized according to Eq. 5.23 and 5.24, are illustrated in Fig. 5.15, 5.16 and 5.17, respectively.
The statistical elaboration of these data allowed to evaluate the statistical parameters, $\mu^\star$, $\sigma^\star$, $COV^\star$, concisely reported in Table 3, clearly showing that the uncertainty of the mean value is represented by a COV around 0.056, while the uncertainties of both the standard deviation and the COV are represented by a COV around 0.112, which could be then adopted as reference values in

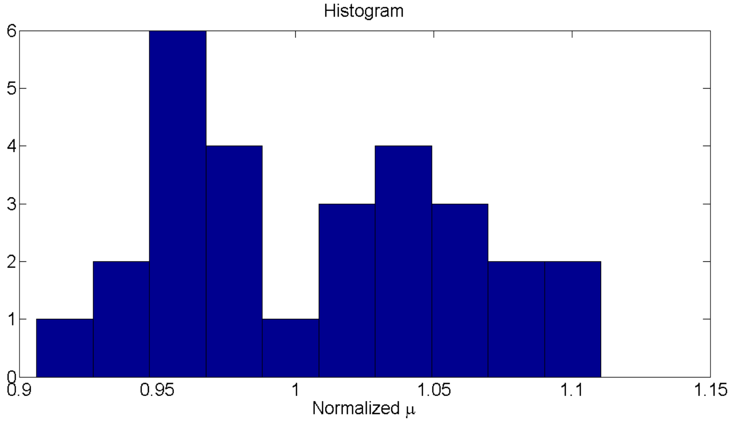

Figure 5.15: Histogram of μ''.

reliability analysis.

5.3.7 Further discussion and validation of the results

In order to further discuss and validate the results, a cluster analysis based on GMM has been also performed considering the global histogram, previously illustrated in Fig. 5.6. As underlined before, the histogram appears smoother compared to those of the annual results, and sub-models could not be identified simply looking at the plot. Besides, the histogram is sensibly skew,

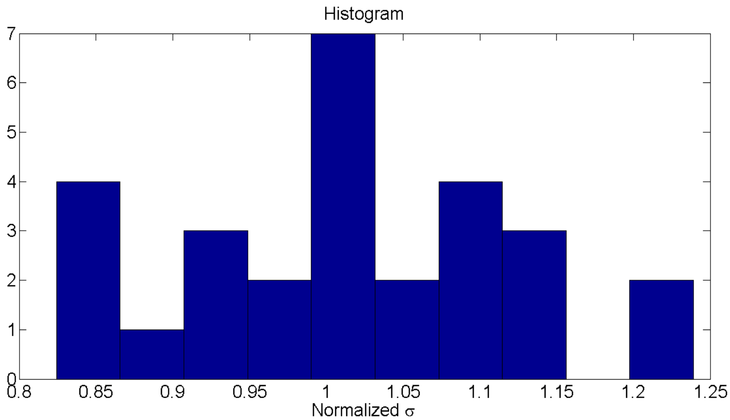

Figure 5.16: Histogram of σ''

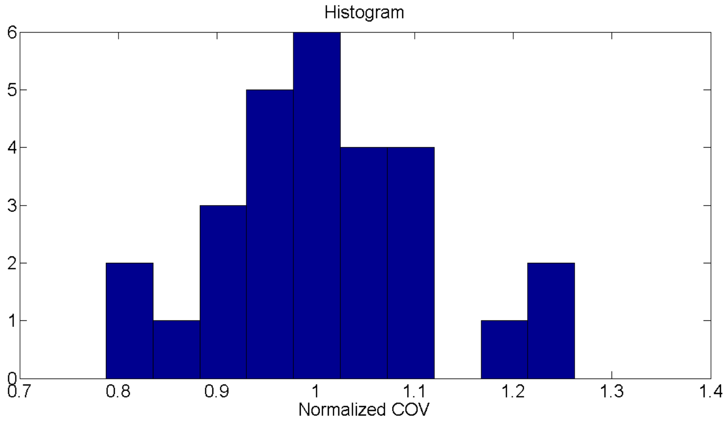

Figure 5.17: Histogram of COV''

	μ''	σ''	COV''
$\mu^{\star}$	1.005	1.010	1.011
$\sigma^{\star}$	0.056	0.113	0.113
$COV^{\star}$	0.056	0.112	0.112

Table 5.2: Statistical parameters of normalized mean value, μ'', standard deviation, σ'', and COV, COV'', of concrete resistance.

with a significant right tail. The EM algorithm was run again considering eight components and random initial conditions, sampled according to the procedure explained in 5.3.4, but considering 5000 simulations. AIC and BIC values were computed for each fitted GMM, and an average model was found according to Eq. 5.20 and 5.21. The GMM so obtained is shown in Fig. 5.18, where also the normal distribution (in blue) and the lognormal distribution (in green) best fitting the global histogram are shown. The normal distribution is characterized by $\mu = 37.86\ N/mm^2$, $\sigma = 16.29$ and $COV = 0.43$; while the lognormal distribution is characterized by $\mu = 38.25\ N/mm^2$, $\sigma = 19.12$ and $COV = 0.50$, so showing both an unrealistically high COV. This result is not surprising, because it confirms once more that analyses of data could not be performed disregarding the identification of their homogeneous populations.
The statistical parameters characterizing each *pdf* of the mixture model and

K	EM + K-means		EM (all data)		
	μ $[N/mm^2]$	COV	μ $[N/mm^2]$	COV	PC
1	15.8	0.239	14.7	0.271	0.1007
2	22.8	0.174	22.9	0.171	0.1779
3	32.4	0.127	31.8	0.132	0.2295
4	40.2	0.089	40.6	0.103	0.2005
5	48.3	0.097	49.9	0.087	0.1409
6	58.9	0.073	59.9	0.076	0.0927
7	67.9	0.074	70.2	0.078	0.0340
8			79.5	0.128	0.0238

Table 5.3: Average statistical parameters of concrete classes obtained applying the EM + k-means algorithms on yearly data and the EM algorithm on the all data.

the percentage of data belonging to each cluster are reported in Table 5.3, where the cells pertaining to the first cluster and to the eight cluster, which were discarded as not representative, have a grey background.
The results match one more time with those previously obtained, both considering annual data (Chapter 5.3.4) and k-means algorithm (Chapter 5.3.5); in fact, the standard deviation is practically independent of the resistance, being mostly in the range $4.0 - 4.5$ MPa, while the COV is decreasing with the resistance in the range $0.08 - 0.18$ except for concrete characterized by a mean value of cubic strength less than 20.0 MPa.

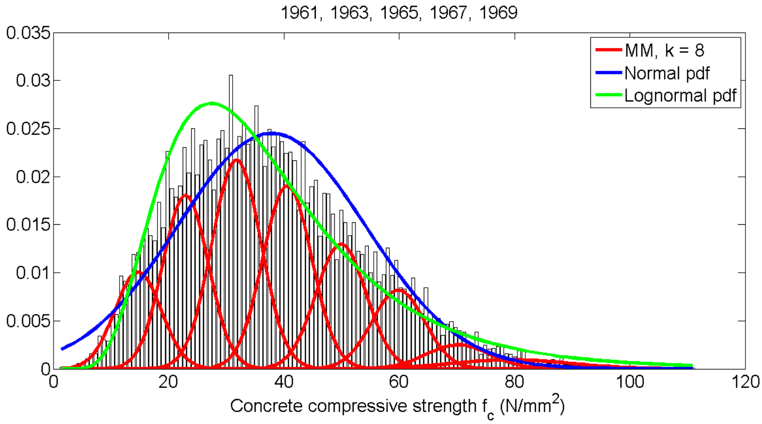

Figure 5.18: Histogram of the amount of cubic strength collected and fitted normal, lognormal and mixture distributions.

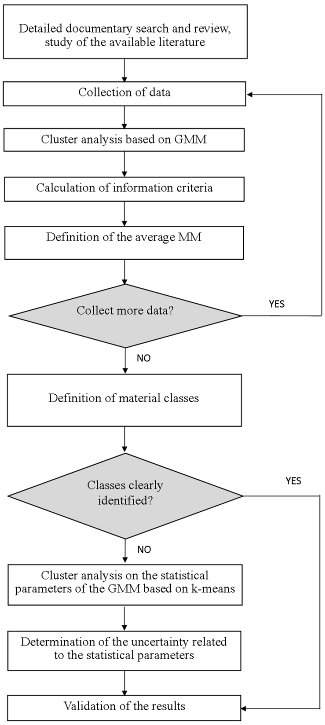

Figure 5.19: General methodology for the identification of concrete classes.

5.4 Overview of the method

The main steps of the method, as outlined in the flowchart illustrated in Fig. 5.19, can be summarized as follows:

- test results, not necessarily associated with some populations or resistance classes, are collected, even coming from several sources;
- data covering a suitable time interval are suitably split into individual databases;
- cluster analyses based on GMM are performed for each database so that homogeneous groups of data and their relevant statistical parameters are properly identified;
- if needed, new data are collected and the analyses are repeated;
- cluster analysis based on k-means is carried out, focusing on the statistical parameters obtained by fitting the Gaussian distributions, to define some reference classes or to achieve additional information;
- uncertainties affecting the statistical parameters are defined, comparing results obtained for each individual database, suitably considering the variations of the statistical parameters from year to year;
- results are further discussed and validated performing a new cluster analysis, based on GMM on the whole database, and final conclusions are drawn.

5.5 Summary

An innovative method has been presented for the statistical analysis of massive sets of secondary test data, where data regarding several discrete resistance classes are mixed together arbitrarily and even collected from several sources. The method allows partitioning the data, identifying clusters of homogenous populations to which they belong so that statistical parameters of each cluster can be assessed. The proposed methodology is extremely general and it allows to elaborate secondary experimental data regarding mechanical properties of each building material characterized by discrete resistance classes, such as concrete, steel, timber, masonry; in particular when reference values of standard deviation and COV are concerned in the analysis. Since the identification of homogeneous populations is performed blindly, the procedure is objective and does not require subjective information, like pre-classification of data, that could be vague or wrong and, anyhow, frequently misleading. The method is very helpful in reliability assessments of existing buildings, where the estimate of COV of relevant material properties is very difficult because direct information concerning the examined structure or primary experimental data derived from acceptance tests or from in-situ investigations are missing or so limited to hinder dependable statistical elaborations. To demonstrate its practical application, the procedure has been applied to a relevant case study, concerning the evaluation of statistical parameters of the cubic strength of the Italian concrete production during the 1960s. The procedure effectively enabled to recognize homogeneous concrete classes and to estimate their statistical properties. Results reveal that in the 1960s, the Italian concrete production was characterized by a standard deviation of the cubic strength of about $4.0 - 4.5$ MPa, practically independent of the strength; the COV thus decreases with the strength and varies in the range $0.25 - 0.06$, that shrinks to $0.17 - 0.06$ for concretes characterized by a mean value of the cubic strength bigger than 20.0 MPa. Those results are in good agreement with the most pertinent conclusions of the relevant literature on the topic. The method also allowed to evaluate the model uncertainties for the mean value of resistance, which is characterized by a COV of about 0.056, and for the standard deviation and the COV, which are both characterized by a COV of about 0.112. Since the results obtained with different approaches agree satisfactorily, it can be concluded that the proposed procedure is not only suitable for the intended applications as long as experimental data are available, but it is also “robust” enough. Finally, as the proposed method was able to identify homogeneous populations and statistical parameters independently on subjective information regarding assignment of data to a particular population or class, it can be applied to analyze any arbitrary mixture of discrete populations, even if the percentage of each of them is unknown, provided that their *pdf*s can be approximated by normal distributions.

CHAPTER 6

Seismic reliability assessment of a concrete water tank based on the Bayesian updating of the finite element model

In this chapter, a method for the reliability assessment of a complex structure that requires FEM analysis is defined. The method is discussed with special reference to a relevant case study: a concrete water tank from the 1960s'. Special attention is devoted to the reliability assessment of the tank under seismic loads, based on a structural identification approach. The calibration of the FEM of the structure is carried out on probabilistic bases, applying Bayes' Theorem and response surface methods. With this case study, it is demonstrated that information regarding the global structural behaviour and local checks can be effectively combined in structural reliability assessments.

Contents

6.1 Introduction

Water tanks and water towers require special attention not only for their fundamental function within more extended and complex network but also because they often represent relevant landmark and architectural heritage. A reliability assessment that considers the actual action and characteristics of the structure, as well as the related uncertainty, represents an essential step for the safe use of those structures and for planning future interventions [84, 158, 44]. This need is especially stringent in seismic engineering, where the presence of several sources of uncertainty – from the intensity of resulting ground motions to load effects caused in the building, as well as the resistance of the structural elements – compel us to use probabilistic reliability assessment in order to make a quantitative evaluation of the structural safety [37]. The limit state according which the probability of failure is computed has to be an explicit function of the basic random variables that affect the reliability. However, this condition is only verified for simple problems or when approximations are used. In order to analyse the structural behaviour of more complex structures, numerical procedures such as those involving FEM calculation are necessary. In this case, the functions of mechanical and physical transformations are often too difficult and the distributions of the load effects cannot be derived from the knowledge of the distributions of the basic variables. In those circumstances, the branch of the performance function related to action effects has an implicit formulation; but, in order to assess the reliability of the structure, efforts should be made to make the function explicit [100]. An option to make the function explicit could be to quantify the uncertainty on the output: in this case, adoption of MC procedure is generally not recommended because it requires a huge number of simulations of the model; more efficient methods are indeed those based on response surface [63, 157]. A response surface has the advantage to be fast to evaluate, requiring a limited set of observations, which are a collection of input/output pairs. Another option is to reduce the uncertainty on the input in such a way that the uncertainty on the output can be disregarded. A non-destructive approach is to establish a parametric correlation between the response characteristics predicted by the models and analogous quantities derived by experimental measures or, in other words, by structural identification procedures [28]. The calibration of the input parameters can be then carried out applying Bayes' theorem [6]. In the present work, the reliability assessment of a concrete water tank under seismic loads is presented. The study involves the definition and the analysis of the FEM of the structure, which is characterised by random inputs. The response of the structure to static and dynamic loads has been measured in order to update input random variables simultaneously. Procedures based on a functional approximation of the system response with gPCE have been applied in order to solve the Bayesian Inverse Problem. The identified model has been used in order to perform the reliability assessment and define the seismic fragility curves for different intensity levels of the Peak Ground Acceleration (PGA) corresponding to the two extreme conditions of empty and full tank.

6.2 General methodology

The main objective of this study is to develop a general probabilistic framework in which structural identification procedures, based on surrogate models and measurements of the global system response, are integrated with local assessment of structural reliability. The proposed methodology, which is based on response surface methods and Bayesian updating, is summarized in the flowchart reported in Fig. 6.1, and can be applied to the analysis of every existing structures, whatever the building material is. The procedure should be seen in the wider perspective of a smarter usage of infrastructure systems, based on feedback loop of data providing evidence for informed decision making not only at the reliability assessment stage but also during design and maintenance planning.

6.2.1 Analytical or finite element modeling of the structure

After the early investigation, a theoretical model of the structure can be set up and a structural analysis aimed at revealing static and dynamic properties, stress patterns and state of damage can be carried out. Often, especially, when the structure is simple, the model can be analytic, but in most cases, numerical models are adopted to investigate the structural behaviour. As it has been already said in Chapter 2 and 4, FEM analysis is by nature a deterministic procedure for determining the response of the structure under specified load condition. If an uncertainty regarding the representative average quantity of a FEM input parameter exists, it corresponds to a lack of knowledge regarding which value better describes the actual behaviour of the structure.

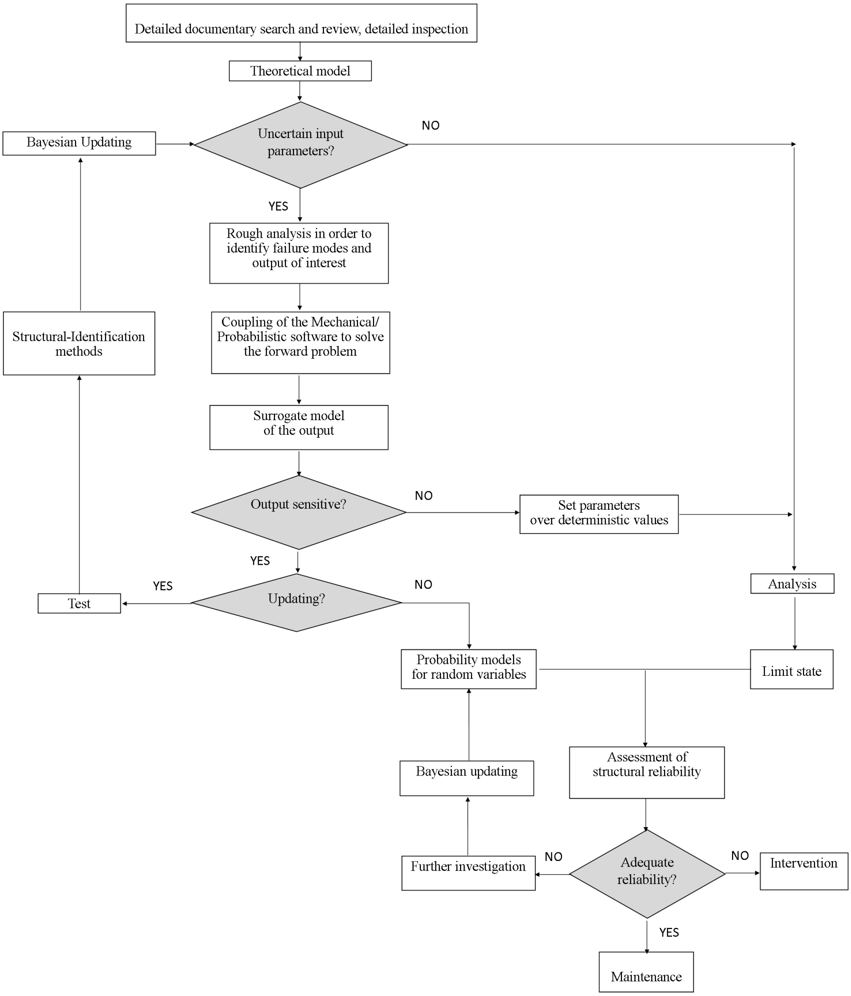

Figure 6.1: General methodology for reliability assessment of existing structures based on response surface methods and Bayesian updating.

The classical approach for uncertainty modelling is to appoint *pdf*s to uncertain quantities, that in this case, they will represent the degree of belief that the input parameters have certain values. The same input parameters might also be affected by aleatoric uncertainty due to an inherent randomness over the structure or the portion of structure considered [77, 124]: however, this second type of uncertainty cannot be captured with traditional FEM. In order to consider also the aleatoric quota, stochastic FEM – an intrusive method in which uncertainty is expressed explicitly within the analysis of the structure, and it is outside of the scope of this work – is required [75, 154]. Preliminary investigations have the objective of limiting the uncertainty regarding input parameters to acceptable ranges with respect to the scope of the analysis. If despite the investigation, input parameters are still affected by uncertainties, these uncertainties should be properly taken into account in the analysis. In this phase, a rough analysis of the structure should be carried out, starting from expected values of input parameters, and possible failure modes should be identified, as well as the relevant structural response parameters to be considered in each assessment steps, like, for example, stresses, internal forces, deformations, displacements and so on. After this, a more detailed study is to be carried out, in which the uncertainty that affects the parameters of the structural response should be quantified. As just said, in case of FEM calculation, the branch of the performance function related to action effects has, in general, an implicit formulation. In order to link explicitly the *pdf*s of load effect and the *pdf*s of the basic variables, it is possible to couple the mechanical model of the structure with a probabilistic code, in order to build a surrogate model of the system response. The response surface is defined by the outcomes of a suitable series of experiments, which consist in considering appropriate values of the input parameters for which the FEM analysis is run [157]. Reducing the uncertainty affecting the input parameters through a structural identification procedure may represent an alternative procedure to make explicit the performance function [160, 111]. By measuring the response of the structure, it is possible to update the input parameters in such a way that theoretical and experimental outcomes will correspond. Measured outputs are generally responses to static loads, such as forces, strains, displacements [130], responses to vibration based experiments like eigenfrequencies, eigenmodes, damping ratios [71], or both [140, 172, 177]. Of course, in the dynamic identification, the output can be directly measured (for example in case of acceleration) or it can be derived (such as frequency response functions or modal data)[148]. Subsequently, the inverse problem is solved applying a

Bayesian inference approach: prior probabilistic models, which are constructed based on the a priori available information, can be transformed into posterior models, using the available experimental data and the probabilistic model for the measurement error. It must be stressed that, in light of the reliability assessment that it will be later carried out, the strategy based on structural identification seems preferable. A huge amount of literature is available on Bayesian inference where reference works in the framework of model updating were developed [11, 96, 159, 20, 93]. The most popular algorithm for obtaining the posterior distribution is probably represented by MCMC, although in the last years some alternatives procedures have been proposed, such those proposed in Chapter 4.

6.3 Case study

6.3.1 The building

The proposed methodology has been applied to a relevant case study. The structure to be assessed is a reinforced concrete water tank built in the early 1960s in the nearby of Pisa (Italy) (Fig. 6.2). The tank is cylindrical, with an average diameter of 9.00 m, a height of 7.50 m and a capacity of about 400 m^3. It is supported by a set of eight inclined columns, arranged at the vertices of an octagon, and connected by three reinforced concrete rings located at different heights. The total height of the structure from the ground level is more than 29 m. Nowadays the structure is used as a water storage for fire protection. In a previous work [35], the structure was verified with deterministic methods. It turned out that the building is safe against dead load and wind action, but it is vulnerable with respect to earthquakes since the seismic demand at the bottom of the column exceeds the resistance. A more refined and accurate evaluation of the structural reliability is then necessary, also in order to plan future interventions for increasing the reliability of the water tank.

6.3.2 Documentary search and early investigation

The documentation available at the beginning of experimental activities consisted of a schematic drawing in which the morphology of the structure and the overall dimensions were shown.It emerged that the foundation consists of

Figure 6.2: The water tank structure considered in the case study.

a reinforced concrete circular plate with a diameter of 11 m; however, there was no indication regarding the depth of the foundation itself. The structure appears very similar to the model of water tank described in [13], that probably was taken as the reference manual by the designer. In order to set up the FEM of the structure further investigations were carried out. The geometric survey demonstrated that columns are characterised by a square section, whose side is 0.56 m, while the cross section of connecting beams is rectangular, 0.56 x 0.30 m. About the reinforcement, a covermeter inspection revealed that:

- the reinforcement of each column is characterised by 8 plain bars (18 mm diameter), while the stirrups are 6 mm plain bars, spaced about 0.20 m; the overlapping joints of rebars were located at mid-height between the rings;

- in the connecting rings, it was possible to detect only the reinforcement of the lower face, consisting of 4 longitudinal bars (18 mm diameter) and 6 mm stirrups, spaced $0.22 - 0.24$ m;

- many bars lack the concrete cover because of corrosion phenomena.

In order to estimate the mechanical characteristics of concrete, measures of the rebound index and of the ultrasonic velocity were performed along the structure, according to the so-called SonReb technique. Several expressions can be found in literature in order to estimate the compressive strength. For a historical review and new perspectives on this topic, please refer to [22]. In [83] a regression formula is proposed, according to which a probabilistic model for the concrete compressive strength can be directly derived. However, most of those researchers have been carried out considering laboratory specimens rather than field study on existing structures. For this reason, the expressions presented in [12] are adopted in the present work, since they have been developed considering the results of several investigation campaigns accomplished with similar measurement devices on existing buildings, aged from one to 50 years, located in the nearby of Pisa. An average value of approximately 60 MPa, with a confidence interval of ± 15 MPa, has been finally considered for the concrete strength f_c. In order to estimate the concrete elastic modulus E, the expression 4.41 has been taken into account. In this way, the concrete elastic modulus has been estimated to be around 38 GPa.

6.3.3 Structural modelling

A FEM of the structure has been made using the finite element program SAP2000 v.14 [139] considering the following assumptions:

- linear elastic constitutive laws for structural materials;
- negligible second-order effects arising from geometric nonlinearity of the structure;
- stiffness of reinforced concrete columns and rings in no crack condition;
- membrane and bending behaviour and no crack condition for the shell elements used to model the cylindrical wall of the tank, lower dome and tank cover;
- Winkler modulus for the soil foundation $k = \infty$.

A linear analysis is carried out in order to simulate the behaviour of the structure during the experimental campaign. This choice is justified by the fact that the applied loads during the test are very low, in the range of linear structural responses. Furthermore, it represents the classical approach for the study of those types of structure, as it is suggested in [54]. The analysis is implemented in conjunction with a reduction factor q that takes into account the beneficial effects of energy dissipation due to ductile mechanisms. Reduction factors are mainly based on empirical observations of the behaviour of common structural systems during earthquakes, and on average, they yield acceptable results. The preliminary deterministic verification of the structure revealed that critical elements are represented by the columns. For this reason, attention is mainly focused on the state of stress and the verification of those structural elements. Furthermore, it also showed that the structure top displacement is of the order of few centimetres: consequently, the $P - \delta$ effects can be reasonably disregarded. The finite element mesh of the structure, characterized by more than 20000 degrees of freedom, is reported in Fig. 6.3: the walls of the reservoir have been modelled using 3D shell elements, while the columns and the connecting ring beams have been modelled using 3D frame elements. Two relevant parameters are considered as affected by uncertainties: the concrete elastic modulus and the depth of the foundation slab with respect to the ground level. Normal distributions have been chosen for both random variables since the probability of the occurrence of very high or very low values is anyhow negligible. For the normal distribution of concrete elastic modulus a mean value $\mu_E^{'} = 40$ GPa and a standard deviation $\sigma_E^{'} = 5$ GPa were assumed; for the normal distribution of the depth of the foundation slab with respect to the ground level $\mu_f^{'} = -1\ m$ and $\sigma_E^{'} = 0.2\ m$ were assumed. As known, two elastic moduli can be assigned for concrete: static and dynamic. Clearly, in dynamic conditions, when negligible stresses are applied, the dynamic modulus of elasticity E_d should be adopted. Since E_d refers to almost purely elastic effects, it corresponds to the initial tangent modulus determined in the static test, while the static modulus E_s is conventionally defined as the secant modulus in the stress range $0 - 0.4\ f_c$. The ratio c between the static elastic modulus and the dynamic elastic modulus, which is always smaller than one, generally increases with the concrete strength and with the concrete age [119]. However the mix composition of concrete significantly affects this ratio: for example, in [102] it is demonstrated that for very low quality aggregates E_s is much lower that E_d; in [92] it is found that difference between E_s and E_d varies in the range $5\% - 20\%$ depending on the type of aggregates; in [173] it is suggested an expression to evaluate E_d, starting from the knowledge of the density and the Poisson ratio of concrete, from which it results that E_d is on average 30% greater than the static modulus. Obviously, a unique

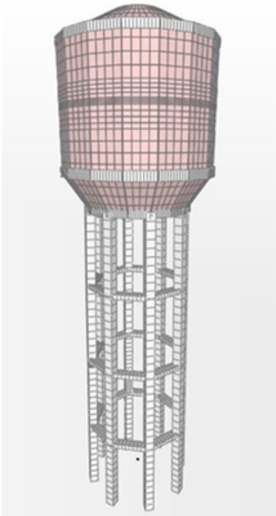

Figure 6.3: The FEM of the structure.

relationship between E_s and E_d does not exist and [119] suggests several alternative expressions. Moreover, in [176], where the adoption of $E_d \approx 1.15E_s$ is suggested, it is concluded that an underestimation of E_d may lead to an underestimation of the effects of seismic actions around 20%.
Although in case of seismic action the structure is subjected to higher stresses compared to those related to a purely elastic behaviour, and it appears more reasonable to perform the seismic analysis considering the static elastic modulus rather than the dynamic one, calibrating also the dynamic modulus may serve in order to have major insight of the concrete resistance and sounder calibration of the static modulus. For this reason, also the ratio between static elastic modulus and dynamic elastic modulus has been considered as a random variable that should be identified. Because of the reason listed above, and especially because the concrete mix is unknown, a uniform distribution with a lower bound of 0.70 and an upper bound of 0.85 is chosen. It is expected that the updated distribution will be Gaussian, as that one of the concrete elastic modulus E_s.

6.3.4 Experimental test campaign

A suitable experimental test campaign was planned in order to measure the structural response to static and dynamic loads. Four tests were performed considering the extreme conditions of empty tank (1°, 2° and 3° test), and tank full of water (4° test). In both conditions, the test sequence consisted in applying increasing loads to the structure by means of an inclined steel cable put under tension by a mobile crane (Fig. 6.4), until occurrence of the brittle failure of a calibrated bar interposed in the cable, in such a way that free vibrations of the tank were induced by the sudden release of the tensile load of the cable. During the tests, loads were measured with a load cell previously calibrated and arranged in series with the cable. The breaking loads of the calibrated bars measured in the four tests are reported in Table 6.1. The measuring system of the static test consisted of 2 HBM W20 displacement transducers located along the columns: transducer $N°1$ was located at 8.77 m height from the soil, while transducer $N°2$ was located at 5.70 m height from the soil. The measured signal is digitized by an HBM UPM100 and saved on the computer by HBM Catman 3.1 Professional. The accuracy of the transducers is defined by a relative error of ± 1% [80]. The force-displacement diagrams recorded during the tests showed a roughly linear progression. The measuring system of the dynamic test consisted indeed of 5 HBM B/200 accelerometers located along the columns. The measured signals were digitized by an HBM MGC MagicPlus and saved to the computer by HBM Catman 3.1 Professional. The accuracy of the sensors is defined by a relative error of up to ± 2% [79]. In the four tests, the signals provided by the accelerometers were processed by Fourier analysis; an average value of the frequency at which the maximum amplification is recorded by each sensor in each test is considered as the natural frequency in the direction of the initial displacement. The related uncertainty has been considered to be ± 1%. The displacement recorded by the two transducers, and the natural frequency are also reported in Table 6.1, referring to each test.

6.3.5 Structural Identification

The stochastic inverse problem has been solved through the gPCE-based methods previously described, approaching it in three consecutive stages:

1. solution of the forward problem, so defining the surrogate model;

Figure 6.4: The loading device.

Test	Load [kN]	Displacement δ_1 [mm]	Displacement δ_2 [mm]	Natural frequency ν [Hz]
Empty tank, 1°	29.49	0.74	0.41	1.12
Empty tank, 2°	61.31	1.49	0.85	1.11
Empty tank, 3°	94.23	2.44	1.42	1.10
Full tank, 1°	85.27	/	/	0.80

Table 6.1: Breaking loads of the calibrated bars and results of the static and dynamic tests.

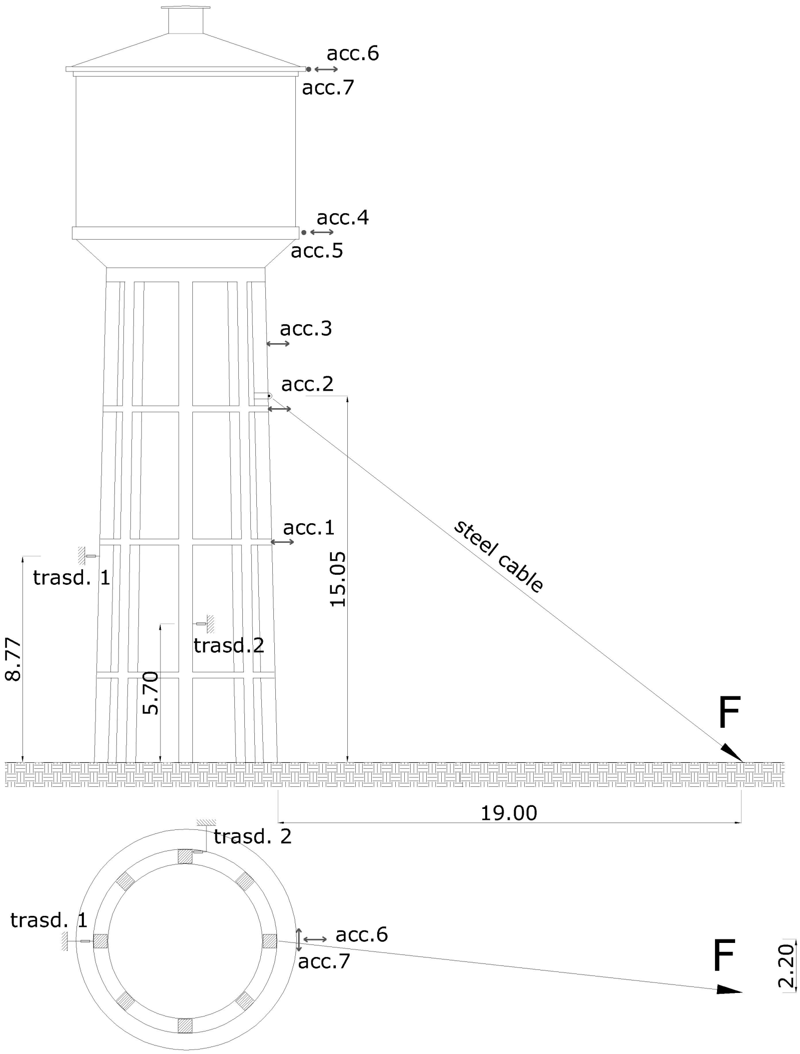

Figure 6.5: Position of transducers and accelerometers along the structure.

2. supervision of the surrogate model, checking the capability of the surrogate model to solve the forward problem;

3. updating of the prior distribution of the input parameters.

An open source stochastic library that works as a toolbox of MATLAB, so-called SGLIB, has been implemented [174]. The FEM software has been coupled with the stochastic library through the application programming interface (API). The algorithm implemented at stages (1) and (2) is summarized in the following, but for further details please refer to Chapter 4.

6.3.5.1 Solution of the forward problem

In order to solve the forward problem, the distributions of the input random parameters are first mapped to the germ distributions. For normally distributed variables, the germ distribution is represented by a standard normal distribution, to which Hermite polynomials are associated; for uniformly distributed variables the germ is indeed represented by a standard uniform distribution, to which Legendre polynomials correspond. The coefficients are then evaluated through interpolation: in order to do that, the model has to be simulated in correspondence of the integration points; for a simple example of how the procedure works in practice, please refer to [109]. In this way, the response surfaces corresponding to the displacements recorded by the two transducers and to the natural frequency of the first mode of vibration have been derived.

6.3.5.2 Supervision of the surrogate model

Once defined a response surface, a given "artificial" measurement is simulated with both the FEM and the surrogate model. The order of the expansion is increased until the two measured values correspond. According to the study carried out in [110], it is possible to state that an expansion of at least fourth degree is required in order to reset the error. Fig. 6.6 shows the obtained surfaces, that describe the relationships between the input random variables and the outputs of interest: (a) E_s and f from δ_1 ; (b) E_s and f from ν; (c) c and E_s from ν; (d) c and f from ν. The horizontal plane represents the quantity that has been measured: in (a), it corresponds to the displacement measured by the first sensor in the third test, while in (b), (c), and (d) it

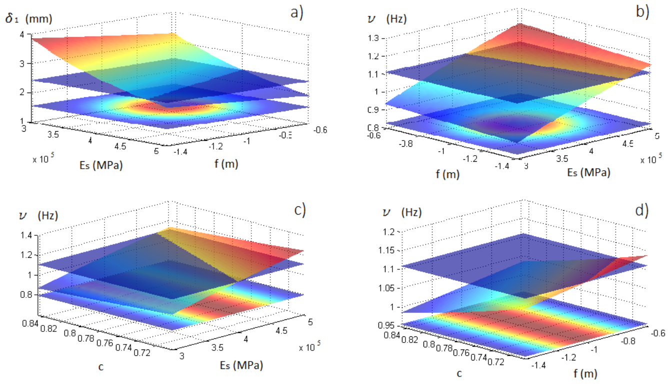

Figure 6.6: Response surfaces for the displacement δ_1 recorder by the first sensor (a) and for the frequency ν (b, c, d)

corresponds to the average value of the frequency obtained during each test. As it was expected, the response surfaces of Fig. 6.6 reveal that, in the range of interest, the dependency of the displacement δ_1 is quasi-linear on the foundation depth f and non-linear on the static elastic modulus E_s; while the dependency of the frequency ν is quasi-linear on both f and E_s, and slightly non-linear on the ratio c between the static and the dynamic elastic moduli.

6.3.5.3 Updating of the distribution

In order to update the prior distribution considering the results of the experimental campaign, both the MCMC algorithm and the method based on the MMSE estimator have been applied. The *pdf* for the measurement error of the recorded displacement is characterized by a standard deviation $\sigma_{\varepsilon,\delta} = 0.01$ mm, while the *pdf* for the measurement error of the frequency is characterized by a standard deviation $\sigma_{\varepsilon,\nu} = 0.01$ Hz. Besides the displacement, only the measured frequency has been considered in the updating process, since its measurements are generally more accurate and more affected by the global structural stiffness variation than the mode shapes. Given the functional approximation of the system responses, it has been possible to update f and E with the measurement of both displacements and frequency; clearly, c has been updated taking into account frequency measurements only. The results of the updating are reported in Table 6.2. Fig. 6.7 shows the samples of the posterior distribution obtained with the MCMC, as well as the prior *pdf*. It is

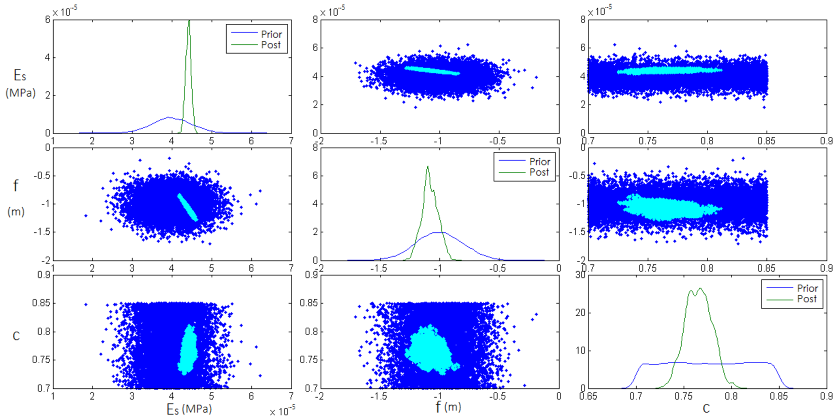

Figure 6.7: Sample points of the posterior *pdf*s, as obtained applying MCMC

	μ'	σ'	COV	μ'' MCMC	σ'' MCMC	COV'' MCMC	μ'' MMSE	σ'' MMSE	COV'' MMSE
f [m]	-1	0.2	0.2	-1.10	0.08	0.07	-1.10	0.07	0.06
E_s [Gpa]	40	5	0.125	44.34	0.85	0.019	44.39	0.68	0.015
c	0.77	0.04	0.05	0.733	0.015	0.19	0.77	0.015	0.019

Table 6.2: Prior and posterior statistics of the input parameter *pdf*s considering MCMC and the method based on the MMSE estimation.

possible to notice that the samples points are aligned along the projection of the intersection between the surface and the plane representing the measurement on the joint space of the variables; the spread of the points along the line depends on the measurement error, while the inclination of the line with respect to the axis determines the magnitude of the updating with respect to each variable.

6.3.6 Determination of actions

6.3.6.1 Static loads

The dead load has been evaluated with respect to the dimension of the structural elements derived from the original drawings and the geometrical surveys. The reinforced concrete unit weight has been estimated to be 25 kN/m^3. The effects of the water inside the tank were evaluated with reference to static and dynamic loading conditions. In dynamic conditions, the action of water has been modelled taking into account the effects that the convective and the impulsive components of the motion of the liquid exert on the walls [65,66]. According to the above-mentioned approach, the total mass of the water inside the tank has been divided into two parts: the impulsive mass, m_i, and the convective mass, m_c. The impulsive mass of the water represents the fraction of the total mass that moves with synchronous motion with respect to the walls of the tank; the convective mass represents the remaining amount of water that moves independently from the tank walls. The two masses can be modelled in a simplified way as concentrated masses, applied in two points, whose heights with respect to the bottom of the tank are suitably assigned. The impulsive mass is considered rigidly linked to the walls of the tank, while the convective mass is considered connected by springs, with elastic constant assigned in function of the characteristics of the liquid and the size of the tank. The values of the two masses, the heights of application and the spring stiffness have been obtained as a function of the size of the tank according to formulae given in [59]. Being $r = 4.50\ m$, the radius of the tank, and $h = 5.04\ m$, the maximum height of the water inside the tank itself, it results:

- $m_i = 244.000\ kg$, the impulsive mass, placed at height $h_i = 3,30\ m$ from the tank bottom;
- $m_c = 135.000\ kg$ the convective mass, placed at height $h_c = 3,60\ m$ from the reservoir bottom.

The resulting stiffness of the springs connecting the convective mass to the wall was determined according to the expression:

$$K_{c_1} = \omega_{c_1}^2 m_{c_1}, \tag{6.1}$$

where ω_{c_1} is the circular frequency and m_{c_1} the participating mass of the water in the tank in the first oscillating mode (sloshing). In the present case, given the size of the tank and the maximum height of the water inside, the elastic constant may be obtained: $K = 619\ kN/m$. This global constraint was modelled by means of 16 radial springs, each with an elastic constant equal to $k = 76.6\ kN/m$, connecting the mass concentrated in the centre with the lateral surface of the tank. Attempt to calibrate m_i, m_c with the measurement of the frequency of the structure in case of full tank was not successful, due to the magnitude of the measurement error (± 1%), that approximately corresponds to the uncertainty in the output due to the uncertainty in the input. For this reason, only a manual heuristic-based calibration procedure of the FEM in case of full tank was applied, in order to check that approximately the theoretical and experimental frequency correspond.

6.3.6.2 Seismic loads

The dynamic linear analysis of the structure has been performed using the modal response spectrum analysis, according to the prescriptions of Eurocode 8 [59, 58]. As the earthquake mainly contributing to seismic hazard is supposed characterized by a surface-wave magnitude, M_s, bigger than 5.5, a type 1 spectrum has been considered [59]. The soil over which the structure stands can be classified as ground type C, whose stratigraphic profile corresponds to deep deposits of dense or medium dense sand, and grave or stiff clay with a thickness from several tens to many hundreds of metres. Obviously, the structural scheme is a typical inverted, since more than 50% of the total mass is placed in the upper third of the height of the structure. In the seismic analysis, the basic value of the behaviour factor, q_0, has been assumed equal to 1.5, as for inverted pendulum belonging to medium ductility class. The responses of all modes of vibration contributing significantly to the global response are taken into account and the effects are combined through the so-called complete quadratic combination:

$$E_E = \sqrt{\sum_i \sum_j \rho_{ij} E_i E_j}, \tag{6.2}$$

where E_E is the seismic action effect under consideration, E_i is the value of this seismic action effect due to the i^{th} mode and ρ_{ij} is the correlation coefficient between the i^{th} mode and the j^{th} mode. Due to the symmetry of the structure, the eccentricity of the storey mass can be disregarded, so that accidental torsional effects have been set to zero.

6.3.7 Verification of the structural reliability

6.3.7.1 Definition of the limit state function

In inverted pendulum system, the dissipation of energy takes place mainly at the base of a single building element. This has also been confirmed for the considered structures by the deterministic analysis previously carried out [35]. For this reason, attention is focused on the action effects induced by seismic actions at the base of the columns. In order to carry out a reliability analysis, a suitable limit state has to be defined. In effect, the reliability assessment of an axially loaded reinforced concrete column could result in a very complex issue, mainly because of two reasons:

- first, the limit state is implicit, since the resisting moment depends on the axial load;
- second, failure depends on the load path.

In the last decades, various approaches have been suggested to tackle the problem. For example, in [51] procedures are proposed in which the load path was determined by load correlation, while in [118, 66] limit states are formulated assuming that loads and resisting moments are not correlated; in both cases, simulation methods are required to generate the strength statistics. In [67] the reliability of short and slender elements considering both the previous cases and performing MC analysis is assessed. In light of the scope of this work, a more rational and convenient approach might be the one embraced in [155, 91], where the conventional reinforced concrete theory is applied, and a limit state is developed considering rectangular stress block parameters; similarly, in [76] a very simple and effective formulations considering the parabola-rectangle diagram for the concrete in compression is developed. The advantage of such assumption is that the resistance moment can be calculated with limited computational effort, through a simplified formulation. Considering, for example, the stress-block diagram for the concrete and the steel yielded in tension, the limit state has the following expression:

$$G = M_R - M_E, \tag{6.3}$$

Parameter	Value
b [mm]	560
h [mm]	560
E_{steel} $[N/mm^2]$	200000

Table 6.3: Deterministic parameters and the relative values.

where M_R is the resistance moment:

$$M_R = \left[0.8 f_c b x (0.5h - 0.4x) + \sigma_s' A_s' (0.5h - d) + f_y A_s ((h - d) - 0.5h)\right] \theta_R, \tag{6.4}$$

and M_E is the acting moment:

$$M_E = M \theta_M. \tag{6.5}$$

In Eq. 6.4 and 6.5 f_c is the concrete resistance, b is width of the section, x is the distance of the neutral axes from the compression side, h is the height of the section, $\sigma_s^{,}$ is the compressive stress in the steel, $A_s^{,}$ is the area of the compressed steel, A_s is the area of the steel in tension, d is the concrete cover, f_y is the yielding strength of the steel, θ_R is the model uncertainty for the resistance moment, θ_M is the model uncertainty for the acting moment. The performance function is thus expressed through a traditional, explicit formulation, which represents the closest approximation to the real limit state function, convenient enough for FORM/SORM reliability analysis. Of course, since x depends on the values of the acting axial load N_E and on the characteristics of the cross section in terms of geometry and material resistances, it has been calculated adopting the average value of each parameter. It must be stressed that, for the purposes of the FEM updating process, M_E, N_E and also x can be assumed characterized by limited uncertainties, when compared to other relevant parameters, so that they can be considered as deterministic quantities, like the dimensions of the elements, b and h, and the steel elastic modulus, E_{steel}, whose values are listed in Table 6.3.

6.3.7.2 Definition of the probability models

The remaining parameters as well as the model uncertainties for the resistance and the load effect have been indeed considered random variables and thus described by appropriate *pdf*s. The type of distribution and the relevant statistical parameters adopted in the case study are listed in Table 6.4. According to the findings of the research described in Chapter 5, the compressive strength of good quality concrete is characterised by low COV. A mean value $\mu_{f_c} = 60\ N/mm^2$ and a standard deviation $\sigma_{f_c} = 10\ N/mm^2$ $(COV = 0.17)$ is considered accordingly. Since it has not been possible to gather any information regarding the yielding strength of the steel, a prior model has been established based on a literature research. In [167], an extensive study of the mechanical properties of the reinforcing bars used in Italy up to '$60s$ has been carried out. Results confirm that the most frequently used steel classes where *Aq42*, like the one originally foreseen for the water tank, and *Aq50*. The yield stress of *Aq42* steel has been assumed lognormally distributed, with a mean value $\mu_{f_y} = 325\ N/mm^2$ and a standard deviation $\sigma_{f_y} = 23\ N/mm^2$. It must be highlighted that adoption of lognormal distribution and similar COV were previously suggested in [52] and [114]. Since in [82] it is suggested that the gamma distribution is the most appropriate to model geometrical quantities, the concrete cover has been assumed to be characterised by a gamma distribution with $\mu_d = 42\ mm$ and $\sigma_d = 5.5\ mm$. The COV regarding the area of the rebar has been assessed to be around the 0.03, as suggested in [90]. Finally, a lognormal distribution, characterized by $\mu_{\theta_R} = 0.95$ and $\sigma_{\theta_R} = 0.13$, has been hypothesized according to [114], for resistance model uncertainty, θ_R, while, following [90], a lognormal distribution, characterized by $\mu_{\theta_E} = 1$ and $\sigma_{\theta_E} = 0.2$ has been adopted for the model uncertainty related to moment and axial load effects in frames, θ_E.

6.3.7.3 Determination of the fragility curves

The limit state developed above enables the estimation of the fragility of the structure to seismic events. The fragility of a structure or a structural component is defined as the conditional probability of failure given demand variables [72]. If the demand variable M_E is an effect of the seismic load, this computational framework is especially called 'seismic fragility assessment'. Therefore seismic fragility is the probability of failure of the structure or

Random variable	Type of distribution	μ	σ	COV
f_c $[N/mm^2]$	Lognormal	60	10	0.17
d [mm]	Gamma	42	5.5	0.13
f_y $[N/mm^2]$	Lognormal	325	23	0.07
A'_s $[cm^2]$	Normal	763	23	0.03
A_s $[cm^2]$	Normal	763	23	0.03
θ_R	Lognormal	0.95	0.13	0.14
θ_M	Lognormal	1	0.2	0.2

Table 6.4: Random variables, types of distribution and relevant statistical parameters adopted in the case study.

structural component given a specified measure of the seismic hazard; different types of measure can be taken into account; however, the most popular one, at least among engineers, is represented by the PGA. Recalling that $G = M_R - M_E$, it holds:

$$p_f = p(G < 0). \tag{6.6}$$

Fragility assessment with respect to the PGA are described in several studies [29, 72, 123, 27], and they can be performed considering different types of analysis: nonlinear time history analysis [95], modal linear analysis [87], or nonlinear static analysis [146]. The assessment of seismic fragility aims not only to better identify the vulnerability of the structure to seismic force, for example in view of planning future interventions, but also to clarify how the uncertainties in the definition of the earthquake magnitude influence the structural response. Fragility assessment has been performed also for the structure under investigation, considering the tank in empty and full condition subject to a series of elastic response spectra, characterized by different intensity levels of the PGA, ranging in the interval $[0.01g - 0.12g]$ with constant increments of 0.01 g. For each level of PGA considered in the analysis, the load effects, M, N and the neutral axis depth x, have been derived, as summarized in Table 6.5 and 6.6, in case of full and empty tank, respectively.

According to the limit states and the probability models previously described, the failure probability and the reliability index β have been derived through

PGA [g]	M [kNm]	N [kN]	x [mm]	PGA [g]	M [kNm]	N [kN]	x [mm]
0.01	25.3	839.6	41.0	0.07	177.5	388.6	30.9
0.02	50.7	761.9	39.1	0.08	202.9	314.0	29.4
0.03	76.1	687.3	37.3	0.09	228.2	239.3	28.0
0.04	101.4	612.6	35.6	0.1	253.6	164.7	26.7
0.05	126.8	538.0	34.0	0.11	278.9	90.0	25.5
0.06	152.1	463.3	32.4	0.12	304.3	15.0	24.3

Table 6.5: Values of the acting moment (M), axial load (N) and neutral axis depth (x) for different values of the PGA (full tank).

PGA [g]	M [kNm]	N [kN]	x [mm]	PGA [g]	M [kNm]	N [kN]	x [mm]
0.01	24.3	388.0	30.9	0.05	121.2	150.3	26.5
0.02	48.5	328.6	29.7	0.06	145.5	90.9	25.5
0.03	72.8	269.2	28.6	0.07	169.7	31.5	24.6
0.04	97.0	209.7	27.5				

Table 6.6: Values of the acting moment (M), axial load (N) and neutral axis depth (x) for different values of the PGA (empty tank).

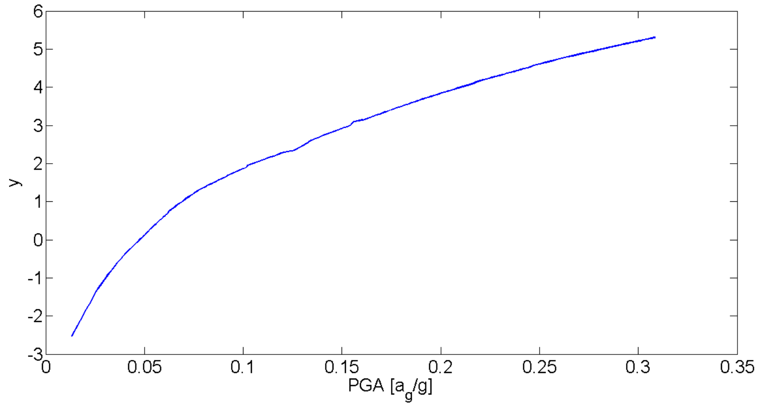

Figure 6.8: CDF of PGA ($T_d = 50$).

a FORM analysis with the software FERUM [19]. The fragility curves $P_f - PGA$ so obtained are reported in Fig. 6.9, in case of full tank, and in Fig. 6.10, in case of empty tank. For a proper clarification of both diagrams, also the cumulative distribution function (CDF) of the PGA is shown in Fig. 6.8, considering a reference period $T_d = 50$ years. The analysis of the results reveals that in the present case study the reliability of the structure under seismic load is generally lower for the empty tank than for full tank. In fact, it should be considered that from one hand the natural period in case of full tank is around 0.9 s, at which corresponds a lower pseudo-acceleration than in case of empty tank, when the natural period is around 1.25 s; on the other hand, the bending resistance of the critical columns decreases for empty tank, because of the lower axial loads acting on them. The fragility assessment for the empty tank has been carried out only for $PGA \leq 0.07$ g, since for higher values of PGA, the column is not subject to compression anymore, but to tension. Moreover, as for $PGA = 0.07$ g the probability of failure is $p_f = 0.35$, it corresponds to an absolutely insufficient level of reliability of the structure. Considering that the target reference reliability index suggested by several Codes [54, 88], for ULSs is $\beta_t = 3.7$ in 50 years, that corresponds to a probability of failure $p_f \approx 1.1 \times 10^{-4}$ in 50 years, and that a structure satisfying this requirement is characterized, under the seismic actions and the seismic combination recommended in [59], by a probability of failure in the range $p_f = 10^{-2} - 10^{-1}$ ($\beta_t = 1.3 - 2.3$) in 50 years, it results that the reliability of the water tank is satisfactory for $PGA < 0.057$ g, when empty, and for $PGA < 0.07$ g, when full. According to [59], the structure should

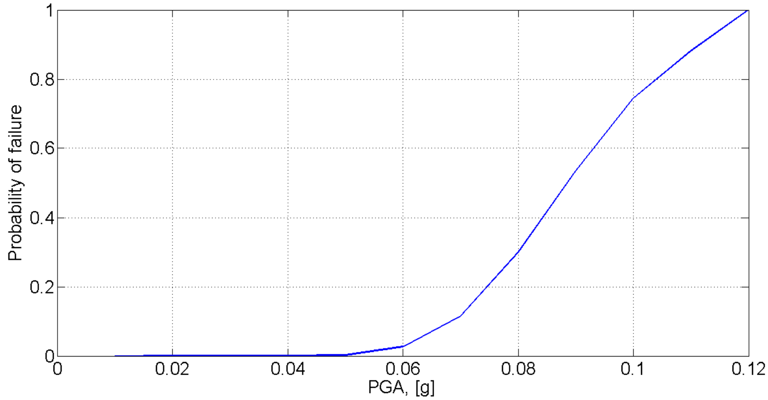

Figure 6.9: Seismic fragility (full tank).

withstand earthquakes characterised by a return period $T_r = 475$ years, 10% probability of exceedance in the reference period $t_d = 50$ years: for the site under consideration a specific study leads to $PGA = 0.118\ g$, which is higher than the limit values determined before, indicating that some intervention is necessary to reduce the seismic vulnerability of the structure. Anyhow, it must be stressed that the return periods corresponding to the above-mentioned limit values of PGA are around 60 years for $PGA = 0.057\ g$ and around 150 years for $PGA = 0.07\ g$. Taking into account that the probability of having full condition is very high, it is possible to state that the tank is able to withstand earthquake characterized by a return period of 100-150 years.

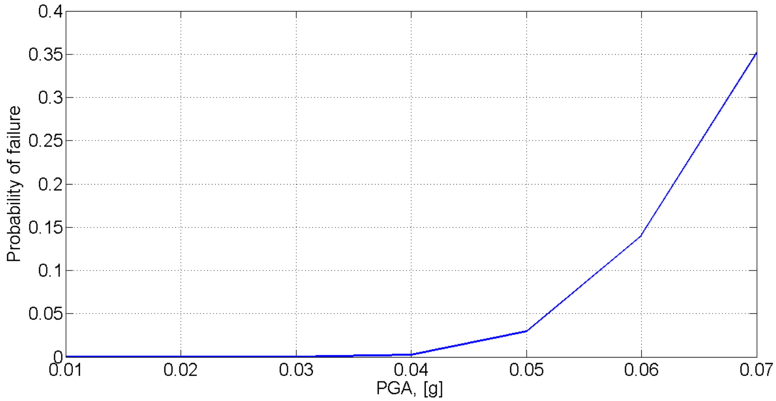

Figure 6.10: Seismic fragility (empty tank).

6.4 Summary

The aim of this Chapter is to illustrate a very promising procedure in which the reliability assessment of the structure has been integrated with global experimental studies regarding the static and dynamic behaviour and structural identification processes based on Bayes' theorem. To demonstrate how promising the proposed method is, the procedure has been applied to assess the reliability of a relevant case study, namely a concrete water tank from the 1960s. Special attention has been devoted to the reliability assessment of the water tank under seismic loads, based on a structural identification approach, where the Bayesian updating is especially focused on the improvement of the finite element model. In defining the relevant limit states to be taken into account, particular attention was devoted to the base end section of the reinforced columns, where plastic hinge formation and dissipation of energy are expected during the earthquake. In the study, it has been specifically considered that the definition of performance function regarding resistance of reinforced concrete sections under bending and axial load is an implicit function and that load effects are affected by uncertainties. In effect, the problem presents a certain complexity mainly for two reasons:

1. first of all, load effects on the structure are calculated through a finite element model: since input parameters are affected by uncertainties, also load effects are. In assessing these uncertainties approximate procedures such as response surface methods have been adopted. However, these

uncertainties are epistemic and therefore should be reduced;

2. the bending resistance of the cross sections depends on the acting axial load.

In order to explore the first aspect, an investigation campaign has been carried out; during the experimental campaign, static and dynamic tests were performed on the structure, measuring displacements, accelerations and relevant modal parameters. The experimental outcomes allowed to set up a structural identification procedure, based on general polynomial chaos expansion of the responses, coupled with the application of Bayes' theorem, aimed to update the input random variables. The originality of the proposed approach, which allows to significantly speed up the structural identification process, also relies upon the simultaneous use of static and dynamic measurements, in order to obtain sounder estimate of the input parameters. To tackle the second aspect, a limit state based on the conventional theory of reinforced concrete has been adopted. In the analysis, load effects have been considered as deterministic parameters after the calibration of the inputs, due to their reduced variability. Finally, fragility assessments of the structure with respect to the peak ground acceleration have been evaluated in empty and full tank conditions. From the fragility assessment, the following conclusion can be derived:

1. the reliability of the structure is not satisfactory considering design earthquake, therefore interventions are necessary to increase it;
2. the empty condition is more severe than the full condition; in effect, at present, the tank is able to withstand earthquakes characterized by a return period of around 60 years when empty, and around 150 years when full;
3. since the probability that the tank is in full condition is very high, it is possible to state that, at present, the tank is able to withstand earthquake characterized by a return period of 100-150 years;
4. anyhow, interventions on the structure are highly recommended in order to improve its performance under seismic actions.

Chapter 7

Reliability assessment of a heritage structure under horizontal loads

In this chapter, a methodology for the probabilistic reliability assessment of heritage buildings is proposed.The procedure addresses investigation and tests on the structure and it considers the use of Bayesian updating techniques for a rational use of the collected information. A practical application of the methodology to a historic aqueduct in Italy is carried out. The main goal is to demonstrate the effectiveness of a probabilistic approach to the reliability assessment of heritage structures.

Contents

7.1 Challenges in applying probabilistic methods to historical buildings

7.1.1 General considerations

It is worth to first ask ourselves what is a historic or heritage building. This term is referred to the ancient structures described in Chapter 2, which in turns comprise not only important monuments but also vernacular heritage mainly made of masonry and wood. Applying probabilistic methods to heritage structures can be particularly difficult for the following reasons:

- they have been often designed according to empirical bases or architectural canons, instead of formal design approach and recognized theories or normative prescription;
- the acquisition of data is a complicated process [143], since the probabilistic description of material properties is affected by great uncertainties, originating for example from the heterogeneity of the material, or its composite nature and anisotropy;
- the identification of transfer functions is commonly difficult, as at least three sources of randomness can be recognized:
    1. actual behaviour is usually represented by means of ideal or simplified models. An example is represented by the de Saint-Venant theory, which considers theoretical restraints and theoretical dimensions, and disregards local effects or stress concentrations, dimensional tolerances and unavoidable errors in executions. This intrinsic source of randomness is also present in new structures, but it has a special impact on existing structures when the classical theory cannot be easily applied;
    2. there is no distinction between decorative and structural elements, so resulting in a strong dependency of different elements and a high degree of structural complexity and indeterminacy;
    3. alteration may have occurred on the structure during the time.

As the structural behaviour depends on a lot of random variables, a probabilistic approach is difficult to govern and the risk that untrustworthy results are obtained can be high. An important type of random variable is here represented by model uncertainty. Model uncertainty is due to idealized mechanical models for the structure and its behaviour, required by practical and operational reasons. It describes random fluctuation of the limit state surface and, although they are taken into account through an additional random variable, it does not have any physical meaning [47]. Model uncertainty is usually assessed through an experimental investigation where model tests are compared with theoretical formulations, relevant case studies of prototypes, information from literature and expert opinion. However, heritage buildings represent single and unique structures, that embody cultural and historic values and for this reason, they cannot be tested to destruction. The assessment of model uncertainty is here a particularly difficult task, that especially requires engineering knowledge and experience in the field.

7.1.2 Main source of model uncertainty: structural alteration

If the structure has experienced several alterations during its life, the uncertainty in the model prediction and thus in the outcome of the reliability assessment could be particularly high. With structural alteration it is meant:

- modification of the structural scheme, even unintentional;
- damage and deterioration processes.

Examples of altered structures are presented in [9] and [86]. Modification and degradation are seldom documented and not easily recognizable, often hidden behind a cosmetic maintenance that could bring to an erroneous perception of the real reliability level. Furthermore, they usually lead to weak points and defects, e.g. heterogeneity, inclusions, voids and cracks, that have a great impact on the structural integrity. As it is shown in [107], disregarding these anomalies entails a misunderstanding of the structural scheme that leads to an unacceptable error in the reliability assessment: therefore, a preliminary careful examination of the structure is necessary in order to understand the actual structural behaviour and to diagnose correctly possible causes of failure. Furthermore, even when previously recognized, structural alterations interplay with other random variables in a complicated way, so that actual transfer functions remain often unknown and difficult to establish with empirical methods. For these reasons, the engineering judgment plays a crucial and

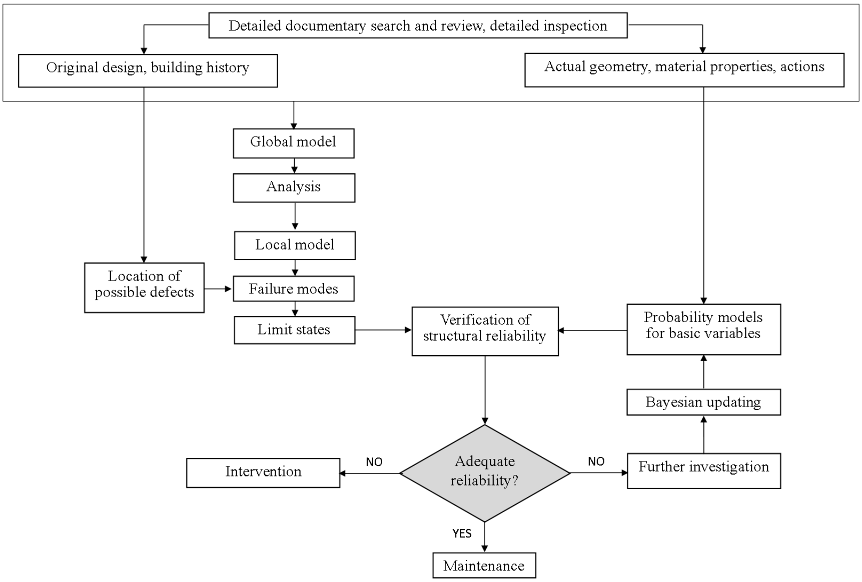

Figure 7.1: Flow chart for planning tests and inspections on historic structures.

essential role in order to evaluate model uncertainties and to effectively include them in a probabilistic reliability analysis.

7.2 Methodology for the planning of investigation and use of information in reliability assessment

7.2.1 General considerations

An ad *ad hoc* flowchart representing a process for planning extensive investigation on historic structures has been developed (Fig. 7.1). The flow chart is an extension of the general flow for the assessment of existing structures presented in [88], when a detailed evaluation is required. The ISO 13822 general flow considers the assessment as a cyclic process, from a preliminary to a more detailed appraisal. In a detailed assessment, investigation of the structure should be carried out; however, no direction for planning investigation is given, e.g. the object of the investigation, what to do with the data collected. The extension would provide the engineer with some indications about these important issues. The process is not compulsory and it has to be interpreted for each individual case. The investigation should be carried out mainly with NDTs, because DTs often endanger the structural integrity of the building and are incompatible with the needs of preservation of cultural, historic and artistic values.

7.2.2 Detailed documentary search and review and detailed inspection

The first step of the procedure is represented by a detailed documentary search and review and a detailed inspection. This step involves archive studies and literature consultations, studies of databases gathering information about comparable structures, historical researches aimed at retrieving information about actions suffered by the structure such as floodings, earthquakes, fire. Detailed inspections are mainly carried out through visual recognition and geometrical surveys. The outcomes of the inspection can be qualitative if they are given in terms of opinions and qualitative judgments, or quantitative if they are represented by a set of values of relevant parameters. As a starting point the original state of the building has to be defined; then the historical evolution of the structure has to be reconstructed and the relevant stages identified. Theoretical studies, like the development of numerical models describing the behaviour of the structure at each stage, should be carried out, if necessary. The goal is to determine causes and effects of damage and degradation processes, as well as motivations of structural modifications and their impact on the structure, in order to identify structural alterations that could affect the assessment. Investigations on existing buildings are also devoted to recognizing the actual structural scheme, relevant actions, material characteristics, geometry and model uncertainties, all information required in order to go further with more complex analysis.

7.2.3 Global model and analysis

At the end of the detailed inspection, a global model of the structure can be set up and structural analyses aimed at revealing static and dynamic properties, stress patterns and the state of damage can be carried out. As many historic buildings are essentially different from modern structures made of steel, reinforced concrete or laminated wood, their analysis cannot be directly performed through modern structural theories of frames, trusses or shells. At the same time, the usual material assumptions of FEM analysis - continue and homogenous ideal models, characterized by definite elastic or non-elastic properties - hardly apply to heterogeneous and non-isotropic material such as masonry and wood. Information about boundary conditions are also limited and imprecise: the foundations of most ancient buildings are usually shallow and affected by noticeable settlements, which are unknown and difficult to measure so that the response of the structure can be substantially altered by slight changes of the soil conditions or sudden actions [85]. A global analysis of the original structure, although rough and approximated, may help in explaining identified damage scenarios and understanding causes of degradation, rather than in defining the action effects in the structural elements, as it has been applied in Chapter 6. It is also possible to gather experimental data about structural characteristics, e.g. performing dynamic identification tests, in order to calibrate the system characteristics or action models until theoretical and experimental information match satisfactorily [7, 34]. A general overview of models and analysis methods of masonry historic building is given in [134]. An interesting approach is also represented by the simulation of the original design. According to this technique, the surviving structure is seen as a historical document that can be 'read' through the use of modern structural analysis or design criteria used when it was built [23, 34]. The outcome of the analysis may be very helpful to understand the original design and dimensioning, to recognize load carrying and transfer mechanisms as well as to explain unusual structural features of the building. Again, buildings conceived with a formal approach can be better studied and analysed compared to ancient structures designed on empirical bases. However, a sound knowledge of construction techniques commonly used in that period may help in identifying the original structural condition and appearance. It should be emphasized that old buildings are frequently structurally inadequate and their construction can be affected by an enormous degree of improvisation. Therefore retracing the original design of the structure helps to address early inspections and recognize defects.

7.2.4 Local model

In addition to the numerical approach, *ad hoc* analysis methods are required for assessing the safety of historic buildings. It should be borne in mind that ancient builders possessed neither knowledge of the resistance of the material nor the underlying mechanics [85]; structures were thus erected according to geometrical rules based on proportions among the structural elements. Furthermore, considered as a homogenized material, masonry exhibits very small tensile strength and high compressive resistance, so that the masonry structure must be mainly subjected to a compressive stress state [122]. According to these considerations, it is possible to state that for ancient buildings the problem of stability is mostly related to the geometry of the structure rather than the resistance of the building materials. Moreover, ancient structures are often affected by extended crack patterns and their structural elements are poorly interconnected: rather than as a unique entity, the structure can be usually represented as a combination of different parts that behave independently, especially if subjected to severe actions, as confirmed by post-earthquake damage surveys [104]. For these reasons, the study of local mechanisms represents an essential step of the safety evaluation of historic masonry buildings. The analysis of local collapse mechanism can be carried out applying different models; for example, a common approach suggested in [121] for the safety evaluation of masonry wall subjected to out-of-plane action is based on the limit analysis, as proposed in [81].

7.2.5 Definition of prior probability models

The acquirable knowledge for the definition of prior probability models has already been largely treated in Chapter 3. It is recalled here that in case of historic structure, poor information is generally available and the engineer has to resort more often to its judgments and knowledge in order to define prior probability model. An exception is often represented by significant and monumental buildings that played an important role in the politics, social interactions, and economics of the day, for which historical records exist [45]. Relevant sources might be treatises, manuals and technical literature published during the 18^{th} and 19^{th} century, collecting in written form the existing empirical knowledge about building practice [50]. For vernacular architecture information is usually even poorer; in this case, a source of knowledge is usually represented by a certain number of contemporary studies and researches about materials, construction techniques, and structural typologies typical of a certain age or geographical area. When literature or modern databases [8] are available for building comparable to those to be assessed, relevant information could be derived by similitude and considered as a prior knowledge pertaining to the actual structure.

7.2.6 Bayesian updating

The improvement of the probabilistic description of resistance parameters may be of interest at first. This step can be achieved through DTs performed in situ or collecting samples for laboratory testing with MDTs. However in case of ancient structures they often entail a loss of the cultural value embodied in the building, therefore they should be avoided. Only in rare cases, destructive test campaigns can be carried out on portions of structures addressed to demolition, or on samples collected in situ (and eventually recomposed in the laboratory) after the collapse of a part of the structure. NDTs may be helpful for refining prior model of resistance parameters; however, the results obtained with those methods may be distorted by several factors, like the high non-homogeneity of the materials and the differences in masonry typologies [18]. More reliable results can be achieved combining several NDTs, but their outcomes will be mainly qualitative unless more precise calibrations are performed on an experimental basis. Obviously, setup of suitable methodologies to extend available experimental results to similar and coeval structures could represent a significant progress in the field. If quantities used as input variables cannot be directly measured but data related to the model response are easier to obtain, a stochastic inverse method may be applied in order to characterize uncertainties regarding input parameters [156]. This approach is especially suitable for historic structures since they represent unique objects whose model uncertainty cannot be defined based on field study. The calculation model can thus be calibrated with the results of monitoring campaign carried out on the actual structure.// An interesting innovative path is here represented by the Bayesian updating of environmental action parameters. In effect, statistical parameters for variable actions are usually established based on simplified and conservative models that take into account data collected in the last decades, when records about earthquakes, floodings and wind have been studied systematically. Random variables are thus defined using conventional statistical approach, incapable of both describing the uncertainty in the estimation of the parameters and dealing with historical information from different sources. In order to derive more realistic and site-specific models that also consider climate change effects as well as historic records, the Bayesian framework represents an interesting and powerful tool; some interesting examples can be found in [169], where a Bayesian approach is applied in order to develop a new Bayesian estimate of the seismic hazard, based on limited earthquake observations and prior seismic hazard studies; in [165], where Bayesian statistics are employed for defining a probabilistic model for the sea levels taking into account also historic data and global warming effects; finally, in [101], where several Bayesian methods applied to the estimation of wind model parameters in the framework of wind energy conversion systems are implemented. The Bayesian updating of environmental actions represents an interesting strategy for obtaining more accurate reliability assessment of existing buildings, but in some cases, it appears overlooked and often too concentrated on updating strength distribution. The setup of appropriate strategies for refinement and improvement of wind, snow and earthquakes probability models could be crucial in order to improve the assessment of existing monumental structures whose cultural value could be endangered not only by interventions but also by destructive test campaigns.

7.2.7 Verification of the structural reliability

The fact that historic structures require *ad hoc* studies depending on analytical mechanically-based models rather than finite element analysis makes FORM/-SORM suitable for the assessment of this type of structures. A direct product of those methods is also represented by sensitivity measures that indicate which variables most influence the result, and therefore may deserve further investigation. It is important to emphasize that weak points and defects should be preliminarily recognized not only to determine model uncertainty but also to identify unpredictable failure mechanisms. The reliability analysis should then consider several limit state formulations for each recognized failure mode. An example is represented in [34], where a historic church struck by an earthquake has been firstly deeply investigated in order to recognize visible and hidden cracks, and then it has been assessed considering the most probable failure mechanisms, deduced by the detected crack pattern. Once the failure probability or the reliability index has been computed, it is compared with the corresponding target values that will serve as a decision criteria. In order to compute the target probability of failure, Eq. 2.15 is considered.

7.3 Case study

7.3.1 The building

In the following a practical application of the flow chart (Fig. 7.1) to a relevant case study is presented. The structure that has been assessed is the Medicean Aqueduct in Pisa, a masonry water work built at the turn of the 16*th* and 17*th* century by the Grand Duke Ferdinando I de' Medici, to convey fresh water from the mountains in the nearby of Pisa to the center of the city (Fig. 7.2). Nowadays the Aqueduct is disused and in a state of decay; in view of its preservation and rehabilitation, a detailed reliability assessment is required.

Figure 7.2: Front of the Medicean Aqueduct.

7.3.2 Detailed documentary search and review and detailed inspection

In-depth visual inspection and geometric surveys have been preliminarily carried out to reveal the actual geometry of the structure, the crack pattern and the material condition. The aqueduct is a 954-spans masonry arch structure characterized by a total length of about 6 km. Each span is from 7 to 4 m tall, 7 m width and 1.2 m depth. The structure is affected by vertical settlements and out-of-plane overturnings, that in some cases determine an inclination up to 9° and provoke cracks especially observed in the key section of arches and in the upper section of pillars. The masonry is characterized by good quality and workmanship, and it is still in good conditions. The material mechanical characteristics should be assessed through destructive in-situ test. However, as it has already been stressed in Chapter 7.2.6, this procedure entails a loss of the cultural value embodied in the building and therefore should be avoided. In order to overcome this limitation, the classification reported in [121] of the most common masonry typologies in Italy and the suggested range of values of the fundamental mechanical parameters are considered. Aleatoric and epistemic uncertainties can be also taken into account, depending on the soundness of the information, adopting suitable confidence factors. Following this criteria, the structure can be classified as undressed stone masonry with regular texture, with a compressive strength of 2.6 MPa and an elastic modulus of 1740 MPa. The historical evolution of the building is traced back comparing maps and pictures collected through a detailed archive study. It was found that the structure was affected by settlements since the construction phase, probably because undersized foundations, while, subsequently, in order to limit the torsional movement affecting the whole structure, buttresses were added every 11 arches. The historical documentation reveals that the structure was struck by two earthquakes which provoked the collapse of several arches, some of which reconstructed later.
With a length of more than 6 km, the Aqueduct stands over soil having different characteristics. A historical documentation reveals that the flat area where the structure has been built was a swampland characterized by a ground level lower than the surrounding plain. Geological surveys have also shown that nowadays the surface is represented by recent layers of alluvial deposits having a variable thickness of $2-5$ m, settled over deposits of gravel. Unfortunately direct surveys of the existing foundation system are not available. According to the construction techniques typical of that period and geographical area, it

has been assumed that most of the pillars have deep foundation characterized by timber piles having a variable length of $6 - 10\ m$. Since the area of the city centre is characterized by over-consolidated soil with improved mechanical characteristics, shallow foundations consisting of rectangular footing are here assumed.
The structure may fail for different circumstances and under several load conditions, which some of them have already been investigated by other authors. In [5] many collapse mechanisms are studied, both analyzing the in-plane and out-of-plane equilibrium of the arcades, also with the help of a FEM analysis of the structure. Considering the forces acting in-plane, equilibrium was satisfied when the centering of the line of thrust was within the transversal section of the arches. Several cases were then considered: deterioration of the masonry of the tympanum due to water seepages; collapse of adjacent arcades; settlements of a column. Results reveal that, in the first case, the line of thrust is nearly centered, such that the sections of arches remain entirely compressed; in the second case the equilibrium with only compression stresses is possible, even with cracking of the sections; in the third situation, equilibrium with only compression stresses is not allowed under certain values of mechanical properties of the ground; this may justify some real situation in which water could penetrate into the ground. Considering the forces acting out-of-plane, a simplified model of an isolated arch has been analyzed, assuming a rigid body on elastic soil and applying several load conditions. Verification with respect to wind action has revealed low tensile stresses and admissible compressions for the masonry. The seismic safety of the structure was expressed in terms of seismic safety index that represents the ratio between the return period of the seismic action that lead to a generic limit state and the corresponding reference return period of the design spectrum of the structure. Three cases have been considered: perfect uprightness of the arcade, initial overhanging and presence of a transverse buttress. Results have pointed out that the structure is already vulnerable to seismic action when it stands in vertical position. Recent studies linked to the following work has been carried in [35]. A global FEM static and dynamic analysis with respect to out of plane actions has been firstly carried out, considering the structure both in its original configuration and in the damaged state. A linear behaviour has been first considered. Regarding the verification of the structure with respect to the seismic action, both the SLS and ULS has been taken into account. The analysis reveals that none of the examined sections satisfies the SLS, and earthquakes with a return period of only 50 years already produce flexural cracking at the bottom of pillars.

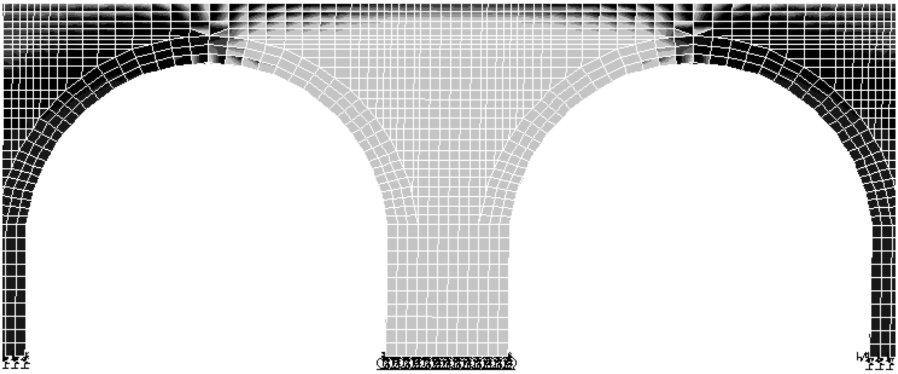

Figure 7.3: Stress pattern of maximum principal stress p_1 due to a vertical settlement of 40 mm and an out-of-plane overturning of 4° in the pillar.

Then a local analysis considering only a pillar and the adjacent arches has been developed, implementing a deformable structural model interacting with soil and considering the second order effects; results confirm the outcome of the linear seismic analysis. Finally, a kinematic analysis has been led on the same macro-element, considering as a possible collapse mechanism the rigid rotation of the structure around a cylindrical hinge located at the pillar's base. The element has been evaluated both in the vertical and inclined position, also considering the presence of a stabilizing force transmitted by the adjacent elements. The outcome shows that the structural element can be considered safe when it stands in a vertical position, while it is near to collapse for an inclination of 9° ; the limit inclination at which the structure can withstand seismic action is assessed around 4° .

7.3.3 Global model and analysis

Since the fundamental aim of this study is to illustrate a practical application of the proposed methodology, the global structural analysis focuses on a significant portion of the Aqueduct composed by eleven arches and bound by two buttresses, located in the nearby of the city centre, in which soil properties, masonry mechanical characteristics and geometry can be considered homogeneous. The clear span of each arch is about 4.3 m tall, 5.3 m width and 1.2 m depth. The total height of the considered portion of the structure is

about 5.2 m; each pillar has a rectangular section, with a length of 1.8 m and a depth of 1.2 m. Starting from the elements closer to the buttresses, each arch presents an increasing inclination of about 1 degree; the central arch is thus affected by the highest inclination of 4°. In order to identify preliminarily the main causes of damage, a linear elastic global FEM analysis has been performed in different condition by means of the FEM software SAP2000 [139]. The arches have been modelled with a mesh of quadrilateral shell elements and the soil has been simulated through elastic spring, according to the Winkler model.
The elastic springs are provided with stiffness towards vertical movement and out-of-plane rotation. In the first case, the parameter has been established according to the characteristic of the soil previously mentioned, and it has a value of $k_z = 20000\ kN/m^3$. In the second case, it has been evaluated by means of a partial model comprising a single pillar realized with SOLID elements over a bed of springs of equal stiffness; the rotational spring's stiffness is evaluated as the ratio between the moment applied at the top of the pillar and the corresponding rotation, and it is assessed around $k_\theta = 2824\ kNm^2/rad$.
The parametric study has been carried out considering self-weight, vertical settlements up to 40 mm and out-of-plane overturning up to 9°. In Fig. 7.3, it is shown the stress pattern in the masonry of the maximum principal stress p_1 due to a settlement of the pillar of 40 mm and an out-of-plane overturning of 4°. The lightest area corresponds to tensile stresses. The stress pattern matches with the commonly detected crack pattern, mainly characterized by vertical cracks in the key section of the arches and horizontal cracks in the pillars, so confirming that one of the most relevant causes of damage is the settlement and overturning of the pillars. Modal parameters, mode shapes and natural frequencies have been determined through a modal analysis, considering damaged conditions as well. As usual, and according to [121], the reduced stiffness in cracked or damaged condition has been simulated halving the elastic modulus of the masonry. The natural frequencies for the first three mode shapes in undamaged and damaged conditions are reported in Table 7.1.

7.3.4 Local model

The Aqueduct represents a series structure, in which the collapse of one unit may provoke the failure of neighbouring structural elements. As the global

	E [MPa]	f_1 [Hz]	f_2 [Hz]	f_3 [Hz]
Undamaged	1740	0.847	1.136	1.563
Damaged	870	0.599	0.806	1.099

Table 7.1: Natural frequencies of the first three mode shapes in damaged and undamaged condition.

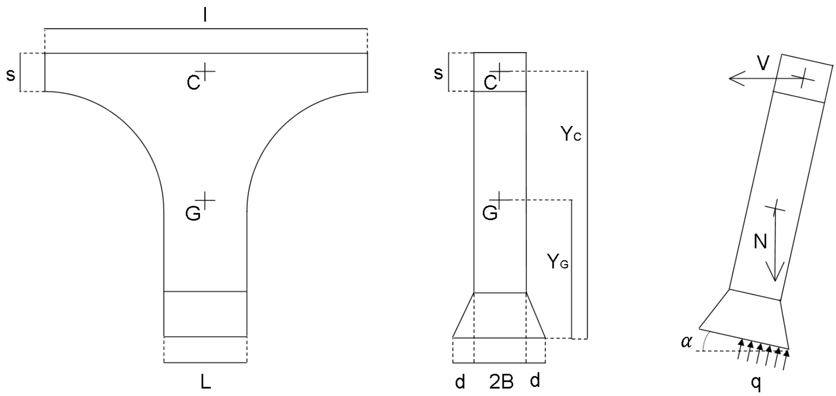

Figure 7.4: Geometry of the local model (for symbols see Tables 2 and 3).

analysis revealed, the key section of the arches may be affected by hidden cracks, therefore a local macro element has been identified and analyzed, composed by a pillar and half portion of each adjacent arch (Fig. 7.4); the symbols are explained in Tables 7.2 and 7.3, where their numerical values are also given. As the height of the considered element is considerably bigger than the dimensions of the pillar and the dimensions of the footing, it can be classified as a tall element. A tall structure standing on deformable soil may undergo a collapse due to a particular form of instability of the equilibrium. Under horizontal actions, the problem may be modelled as that of an inverted pendulum [35], i.e. a weightless rigid shaft with a mass concentrated at the top. The second step of the analysis consists in identifying the failure mechanisms. It is important to bear in mind that structures subjected to horizontal (and especially seismic) forces often show ductile behaviour, that is the ability to redistribute actions among several structural elements; as a consequence, the actual resistance appears to be higher than that calculated through elastic

Figure 7.5: Case 1: the out of plane rotation affects the whole series of arches bound by the buttresses.

Figure 7.6: Case 2: the out of plane rotation affects only the central element.

models. As it has already noticed in [81], although individual stones may be brittle, masonry is, in fact, a ductile material, and particular mechanisms that allow for a redistribution of action may activate. According to these considerations, the following limit situations are considered:

- the out-of-plane rotation affects the whole series of 11 arches bound by two consecutive buttresses (Fig. 7.5);
- the out-of-plane rotation affects only the central element (Fig. 7.6).

In the first case the central element is scarcely affected by the contrasting action provided by the buttresses, therefore the stabilizing force is just represented by the soil resistance while overturning forces are represented by the self-weight and the horizontal action. In the second situation, an arch mechanism may develop in the thickness of the elements, due to the fact that the adjacent arches restrain the rotation of the central element. In the latter case the macro element is subjected to a stabilizing horizontal force V whose intensity depends especially on three factors: 1) the out-of-plane rotation α, represented here by the displacement of a control point δ ; 2) the geometry of the elements involved; 3) the resistance of the masonry. The value of the stabilizing horizontal force V, determined under the hypotheses summarized in Chapter 7.3.5, is:

$$V = \frac{0.04 f_m st(2 - \frac{\delta}{t}(3 - 4\frac{\delta}{t})}{\sqrt{2 - (\frac{\delta}{t})^2 + (\frac{\delta}{t})^2}}. \tag{7.1}$$

In Eq. 7.1, δ is the displacement on the top of the macro-element, equals to $Y_C sin\alpha$, where Y_C is the height of the control point and α is the out-of-plane rotation; f_m is the masonry compressive strength; s is the height of the key section; t is the thickness of the arch and l the length of the macroelement. The geometric parameters and the force originated with the arch mechanism are represented in Fig. 7.7. In Fig. 7.8, the stabilizing horizontal force V is plotted as a function of the displacement δ. Note that the force V reaches a maximum value when 0.1m$< \delta < 0.2$ m, corresponding to $1.2° < \alpha < 2.5°$, and it decreases, until assuming a null value when $\delta > 0.8$ m, corresponding to $\alpha \approx 9.5°$. According to [39], the failure mechanism can be expressed in mathematical terms through the following limit state:

$$G = M_R - M_E, \tag{7.2}$$

where M_R is the stabilizing reacting moment and M_E is the acting moment:

$$M_R = \phi(B + d)N(1 - \frac{N}{N^*})^{\rho}, \tag{7.3}$$

$$N^* = 2q(B + d)(L + 2d). \tag{7.4}$$

In Eq.7.3 ϕ and ρ are dimensionless parameters, whose value have been determined under the assumption that the bearing capacity is uniformly distributed with constant intensity over a part of the surface of the foundation; L is the width of the pillar; d is the lateral enlargement of the pillar at the foundation level; B is half thickness of the pillar; N is the self-weight of the structure; q is the soil resistance. When the structure is not subjected to horizontal actions, the destabilizing moment is only due to the out-of-plane rotation and depends exclusively on the self-weight of the structure:

$$M_E = NY_G sin\alpha, \tag{7.5}$$

being Y_G is the height of the centre of mass.

7.3.5 Definition of probability models

In the following, they are considered as deterministic the parameters ϕ and ρ, included in the inverted pendulum model, and the geometric data that can be easily measured e.g. dimensions of elements, out of plane rotation α, centroid and points of application of forces (Table 7.3). The other relevant parameters are considered random (Table 7.2).
Each random variable is probabilistically described by a prior *pdf* based on detailed documentary search and review and detailed inspection. According to [82], a Gaussian *pdf* is frequently used as a theoretical model of self-weight,

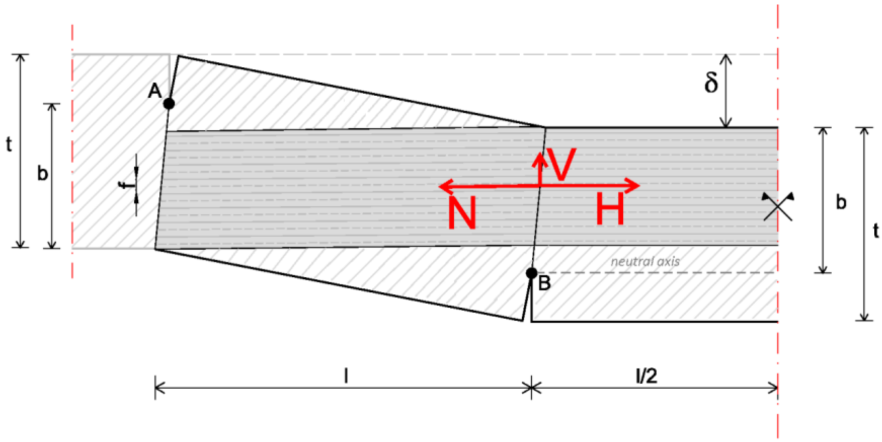

Figure 7.7: The arch mechanism: the adjacent arches restrain the rotation of the central element. In this case the central element is subjected to a stabilizing horizontal forces V.

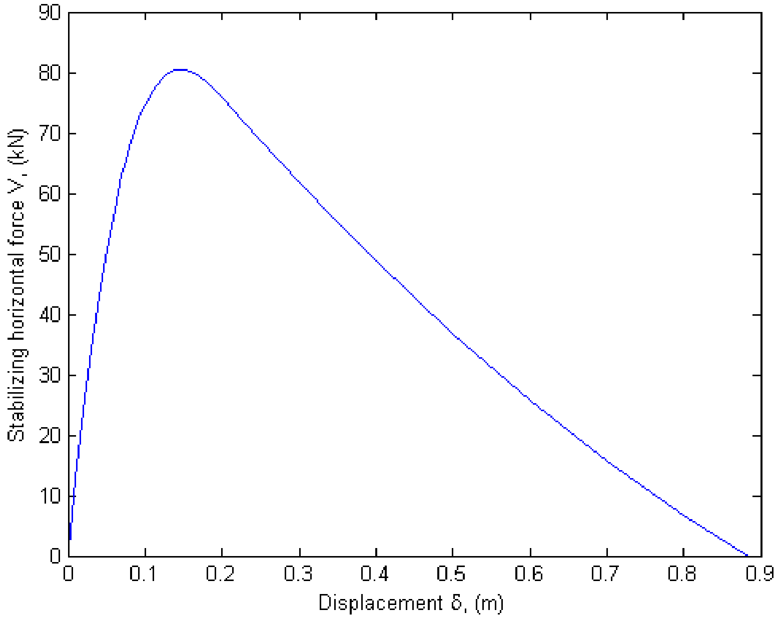

Figure 7.8: The stabilizing horizontal force V plotted as a function of the displacement δ.

strength and geometric properties if $COV < 0.20$ and the skewness $\alpha \approx 0$. Also [90] suggests that a Gaussian *pdf* cannot be rejected for a wide variety of geometric dimension. Therefore a normal probability model is chosen for the self-weight and geometric properties of the foundation. A lognormal distribution is chosen for masonry compressive strength, soil resistance and model uncertainty instead [82].
As it is already explained in Chapter 7.3.2, the Aqueduct is mainly characterized by undressed stone masonry of extremely variant units dimension and appearance. The semicircular arches are made of bricks while the pillars are constituted by stones of greater dimension. The masonry cross section is regularized by thin layers of bricks. The water-logline on the top was made of Cotto tiles, later covered by stone plates. The arcades were built with construction techniques similar to those used in Roman aqueducts. The study carried out in [5] has revealed that the columns have a thickness equal to 1/3 of the clear span, being between 1/2 and 1/5, typical values of the Roman bridges. An extensive analysis of the stones and mortar which characterize the structure has been developed in [68, 69, 132], where the materials' chemical and mineralogical composition, the physical properties and the mechanical characteristics have been assessed. Selciferous limestone is the main stone used to build the undressed stone masonry of the Aqueduct, followed by marble and quarzites from the Monte Pisano, bricks and 'macigno' sandstone. The dimension of the stones varies in the range $20 - 30\ cm$. The source quarries are located near San Giuliano Terme, a locality about five kilometres from Pisa. The vicinity of the quarries to the city centre justifies the fact that this material has been widely implemented in several Medieval buildings, whom the main example is probably represented by the City Walls. The analysis of samples extracted from the quarries reveals that the compressive strength is assessed around $150 - 160\ N/mm^2$; the resistance to alteration is considered very good.
The samples of mortar examined in [132] come from the Aqueduct itself and they are both jointing and filling. The material shows high adhesion to building structures and no shrinking cracks, bubbles and deterioration phenomena visible. Although the measured data of uniaxial compression strengths (average value $16.6 \pm 4.9\ N/mm^2$) are limited by their low accuracy, they appear consistent with the hydraulic nature of the Medicean Aqueduct's mortars; however, clear evidence of hydraulic additives does not exist. Since the properties of the historic masonry reported in [121] match very well with experimental results obtained on the Aqueduct and on similar historical structures, prior *pdf*

of material mechanical properties can be established considering the values proposed by the above mentioned Italian Code. According also to [31], the minimum and maximum values given for each parameter are considered certain fractiles of the underlying *pdf*. The prior *pdf* for the compressive strength f_m is thus characterized by a mean value $\mu_{f_m} = 3.00\ N/mm^2$ and $COV = 0.19$. The assumed mean value is also confirmed by an experimental investigation carried out in [105], aimed at characterizing stone masonry walls with regular texture. For the specific weight of this type of masonry, a Gaussian *pdf* can be adopted with a nominal value of 21 kN/m^3 [31], while according to [143], where a similar historic structure has been studied applying probabilistic method, $COV = 0.1$ is assumed, in order to take into account heterogeneities and voids. These assumptions lead to a Gaussian *pdf* for the self-weight of the macro element, characterized by $\mu_N = 525\ kN$, and $\sigma_N = 52.5$. The portion of the Aqueduct considered in this study is situated in the nearby of the centre of the city. The study of available literature points out that there the structure rests on a layer of homogeneous cohesive soil (mostly silt and clay), characterized by a depth of $3 - 5\ m$. Below the shallow step footing foundation, the soil is over-consolidated. According to geological judgments and a literature review regarding the area of Pisa, the soil bearing resistance in terms of stresses can be assumed around of 600 kPa, with an estimated minim value of 400 kPa and maximum value of 800 kPa [57].

In [126] it is stated that although a soil mass is considered as being homogeneous for geotechnical purposes, it usually shows a variation of properties from one spatial location to another. The inherent variability of soil properties is separated into two factors: a spatial trend and fluctuations in this spatial trend [128]. In order to characterize this model, a reasonable amount of detailed in situ investigation and laboratory tests are required, taking also into account the effects of consolidation. For the sake of this study, whose aim is to illustrate the proposed methodology, averaged values for the soil properties are preliminarily considered. According to [141], and taking into account the estimated minimum and maximum value, a $COV = 0.3$ is assumed for the soil bearing resistance. This relatively high value is justified by the fact that COVs of natural geomaterials are usually larger of artificial building materials such as concrete or masonry [127], and it is supported by [164], where an extensive review of common values and ranges of statistical characteristics of soil properties is provided.

The pillar is characterized by a square section whose area is given by $A = 2BL$, where B, representing the half thickness, and L, representing the width of the

pillar, have been assumed deterministic. The pillar stands on a foundation whose dimensions are larger than the pillar section: the enlargement of the foundation compared to the basis of the pillar is described by an additional geometric parameter d, which depends on the depth of the foundation. The *pdf* that describes d has been assumed normal, characterized by a mean value $\mu_d = 0.40\ m$, and $COV = 0.15$, assessed through engineering judgments.
The uncertainties that arise from the choice of the resistance model can be represented by additional random variables described by a lognormal distribution with unit mean and a suitable standard deviation, multiplying a component of the limit state formulation. In this study, two different kinds of model uncertainty are introduced, one representing the uncertainty in the contribution to the resistance given by the soil, θ_q, the other representing the uncertainty in the contribution to the resistance given by the arch mechanism, θ_V. According to [117], several factors affect the entity of model uncertainty in case of shallow foundations: among the others, modelling of static loads, scale effect, stress condition, anisotropy, strain-softening and progressive failure. In the limit state formulation deduced by the considered model, the soil bearing resistance in terms of stresses is assumed to be uniformly distributed with constant intensity over a part of the surface of the foundation, such that its centre coincides with the point of application of the external load [39]. However, this assumption is hardly verified if the foundation-soil interface presents some heterogeneities, as it is the case with old masonry structure. The model uncertainty factor θ_q for soil resistance takes into account this phenomenon, and, according to the engineer's experience, its COV has been set equal to 0.15. The limit values of the stabilizing force V due to the arch mechanism in the thickness of the elements have been calculated according to the following hypotheses:

1. the constitutive law for masonry is a stress-block;
2. strains are zero in the theoretical external hinges of the end sections of the central element (point A in Fig. 7.7);
3. in each constrained section stresses have equal value;
4. the neutral line in the central element of the arch is rectilinear (see Fig. 7.7).

Due to the heterogeneities of the masonry, these assumptions may be not verified; the variable θ_V takes into account the model uncertainty referred to the stabilizing force V, and its COV has been assumed equal to 0.15, according to

Variable	Symbol	Distribution	μ	COV	Characteristics value
Soil bearing resistance (stresses)	q	Lognormal	600 kPa	0.3	354.6 kPa
Foundation enlargment	d	Normal	0.4 m	0.15	0.3 m
Self-weight of the macroelement	N	Normal	525 kN	0.1	611.4 kN
Masonry compressive strength	f_m	Lognormal	3.0 MPa	0.19	2.2 MPa
Model uncertainty of q	θ_q	Lognormal	1	0.15	/
Model uncertainty of V	θ_V	Lognormal	1	0.15	/

Table 7.2: Random parameters.

engineering judgments. Statistical parameters and *pdf*s of random parameters are summarized in Table 7.2, while values of deterministic parameters are summarized in Table 7.3.

7.3.6 Updating of the wind speed

In agreement with [55], wind actions on buildings can be defined according to a process of macrozonation and microzonation. In the present case, the process of macrozonation consists in collecting data from weather stations scattered over the Italian peninsula; the observations are then analysed and elaborated in order to determine the macro areas represented by the same characteristic value of the basic wind speed $v_{b,0}$.
The wind force is then brought back to a local level through a set of coefficients that takes into account local condition, like exposure, terrain roughness, orography. In this way, it is possible to determine the value of the wind action in each particular location.

Parameter	Symbol	Value
Dimensionless parameter	ϕ	0.50
Dimensionless parameter	ρ	1
Half thickness of the pillar	B	0.60 m
Width of the pillar	L	1.80 m
Height of the center of mass	Y_G	4.61 m
Out-of-plane rotation	α	parameter
Height of the key section	s	0.88 m
Length of the macro element	l	7.12 m
Height of the control point	Y_C	6.61 m
Stabilizing force	V	parameter

Table 7.3: Deterministic parameters.

In the Mediterranean climatic region, extreme values of wind speed are commonly described by a Gumbel *pdf* [55, 54]: the characteristic value of the wind speed is defined as the 98%-fractile on yearly base, namely the probability that it is exceeded in one year is 2%, corresponding to a return period of about 50 years, while COV can be typically assumed equal to 0.2 [55]. The region near Pisa where the structure is located is approximately 10 km far from the Tyrrhenian sea, on the boundary between the coastal area and the hinterland; since those areas are respectively characterized by a characteristic value $v_{b,0} = 28.5$ m/s and $v_{b,0} = 27$ m/s, the characteristic wind speed $v_{b,0}$ assumed in the nearby of the aqueduct is around 27.75 m/s. Based on this estimation is also possible to evaluate the scale κ and location λ parameters of the Gumbel distribution. The following values are finally obtained: $\lambda = 16.63$ m/s and $\kappa = 2.85$ m/s.
Both hyperparameters of the Gumbel distribution are affected by some uncertainty: it is assumed that the scale parameter κ is interested in the representativeness of the population of data used to evaluate the Gumbel distribution, and a $COV = 5\%$ is estimated; the location parameter is indeed influenced by regional characteristics of the wind action, and its uncertainty can be estab-

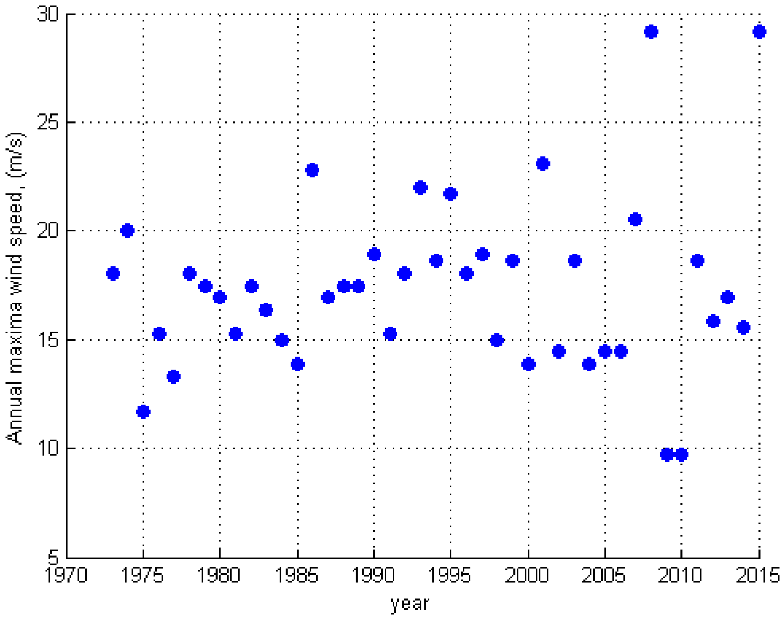

Figure 7.9: Annual maxima of wind speed at weather station of Pisa Airport.

lished based on the uncertainty concerning the characteristic wind speed $v_{b,0}$, that varies in the range $v_{b,0,MIN} = 27\ m/s$ and $v_{b,0,MAX} = 28.5\ m/s$. The hyperparameters are considered uncorrelated. In conclusion, both parameters are described by a Gaussian *pdf* [32] with $\mu_\kappa = 2.85$ m/s and $\sigma_\kappa = 0.143\ m/s$ in case of the scale parameter κ, and $\mu'_\lambda = 16.63\ m/s$ and $\sigma_\lambda = 1.250\ m/s$ in case of the location parameter λ.
The prior distribution for the scale and location parameters of the wind speed can be updated according to annual wind speed observations collected from the weather station of the Pisa Airport, in the nearby of the Aqueduct. Observations have been gathered since 1973, therefore a total number of 43 records are available so far (Fig. 7.9).

In order to make inference on the prior probability model and obtain a posterior probability distribution, the computational technique of MCMC has been implemented [52]. The posterior *pdf* for the location parameter is characterised by a mean value $\mu''_\lambda = 15.43\ m/s$ and a standard deviation $\sigma''_\lambda = 0.432\ m/s$, while for the scale parameter $\mu''_\kappa = 2.96\ m/s$ and $\sigma''_\kappa = 0.125$ m/s are obtained. The updated Gumbel distribution is thus characterized by

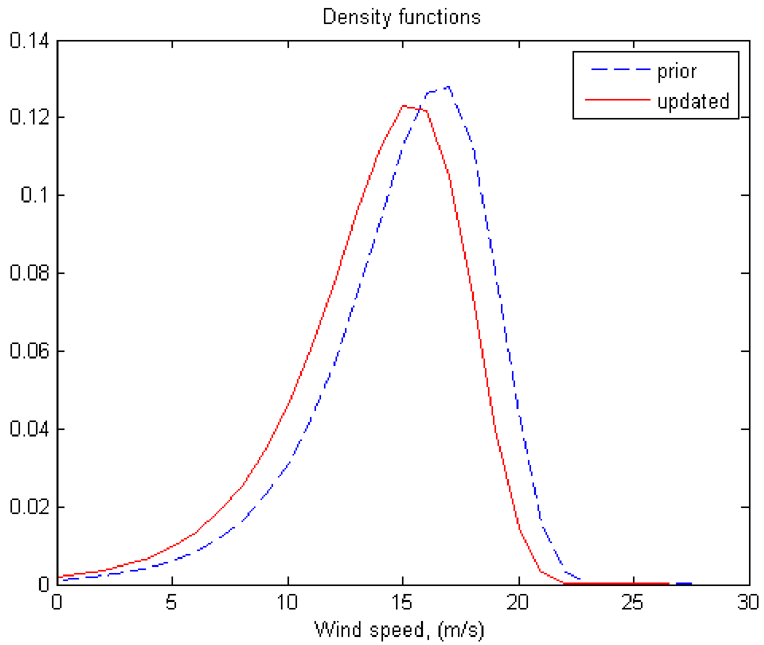

Figure 7.10: Prior and updated Gumbel distributions of basic wind speed (1 year design life).

$\lambda'' = 15.43\ m/s$ and $\kappa'' = 2.96\ m/s$ (Fig. 7.10).

7.3.7 Verification of the structural reliability

7.3.7.1 Definition of the target reliability index

A target probability of failure has been defined based on the formula described by Eq. 7.9 and proposed by [143]. The parameters have been set up considering a design working life $t_d = 50$ years and a social criterion factor $S_C = 0.05$, that is the value usually associated to historic structures of this type; it is assumed that the failure is reached by a gradual deterioration that could be hidden from view, therefore the warning factor is considered equal to 0.3; both the economic factor and the activity factor are evaluated to be 1, while the number of people put to danger in case of failure is esteemed to be 5. The following target probability of failure is obtained:

$$p_f = 10^{-4} 0.05 \frac{50}{5} \frac{1}{0.3} 1 = 1.7 \times 10^{-4}. \tag{7.6}$$

that approximately corresponds to a target reliability index $\beta_t = 3.7$.

7.3.7.2 Verification under wind and seismic force

An analysis has been carried out in order to assess the reliability of the element under wind loads; the following limit state has been considered:

$$G = M_R - M_E, \tag{7.7}$$

$$M_R = \phi(B + d)N(1 - \frac{N}{N^*})^\rho \theta_q + 2V\theta_V Y_C, \tag{7.8}$$

$$M_E = NY_G sen\alpha + WY_G. \tag{7.9}$$

According to [21] and [50], the wind load has been calculated as:

$$W = pA, \tag{7.10}$$

where A is the surface of the structure orthogonal to the wind direction and p is the wind pressure:

$$p = q_b c_e c_p = q_b C_w, \tag{7.11}$$

where c_e is the exposure coefficient and c_p is the shape coefficient, that can be represented by the product C_w. According to [90], the coefficients involved in the assessment of the wind action contain uncertainties, and therefore should be represented by a *pdf*. Each coefficient is described by a lognormal distribution, that for c_e has a mean value $\mu_{c_e} = 1$ and $V_{c_e} = 0.2$ and $\mu_{c_p} = 1$ and $V_{c_p} = 0.2$ for c_p. In conclusion C_w is again lognormal and characterized by $\mu_{C_w} = 1$ and $V_{C_w} = 0.3$ (Table 7.4). The kinetic reference pressure q_b is usually calculated in the following way:

$$q_b = \frac{1}{2} \rho v_{b,0}^2 \tag{7.12}$$

where $\rho = 1.25\ kgm^3$ is the air density and $v_{b,0}$ is the characteristic wind speed, that has been assumed random. Recalling that the design working life considered to derive Eq. 7.8 is 50 years, the prior and updated Gumbel CDFs

Variable	Symbol	Distribution	Ratio $\frac{Expected}{Computed}$	COV
Exposure coefficient	c_e	Lognormal	0.8	0.2
Shape coefficient	c_p	Lognormal	1	0.2
Product of the coefficients	C_w	Lognormal	0.8	0.3

Table 7.4: Statistical properties of wind load coefficients.

$F_{50}(x)$ referring to 50 years have been derived from the annual ones, $F_1(x)$, by means of the usual formula:

$$F_{50}(x) = [\ F_1(x)]\ ^{50}, \tag{7.13}$$

so that they assume the form being $ln(50) \approx 3.912$. The statistical parameter of both the prior and the posterior wind speed *pdf* referring to a reference period $t_r = 50$ years are thus represented by, respectively:

$$\kappa'_{50} = \kappa'_1 = 2.85, \tag{7.14}$$

$$\lambda''_{50} = \lambda''_1 + \kappa''_1 ln(50) = 15.43 + 11.58 = 27.09, \tag{7.15}$$

$$\kappa''_{50} = \kappa''_1 = 2.93. \tag{7.16}$$

A parametric study aimed at determining the critical inclination for which the arch may be affected by failure has been arranged and developed using [19]. The results of the FORM analysis are showed in Fig. 7.11. The highest reliability index is obtained in correspondence of $\alpha \approx 1.7°$, when the stabilizing force assumes the maximum value $V \approx 80.4\ kN$. For $\alpha > 1.7°$, the stabilizing force decreases, and thus the reliability of the structure. Results reveal that the structure is not safe anymore for $\alpha \approx 6.5°$. The reliability of the element has been assessed also considering the seismic force and the ULS. Conversely to Chapter 6, the fragility curve represents here the probability of failure of the structure for different inclination and fixed seismic action. The acting moment in the limit state equation assumes here the following expression:

$$M_E = NY_G sen\alpha + F_S Y_G, \tag{7.17}$$

where $F_S = 86.97\ kN$ is the seismic force consistent with a return period $T_{(R,ULS)} = 475$ years and a design life $t_d = 50$ years. The seismic action has

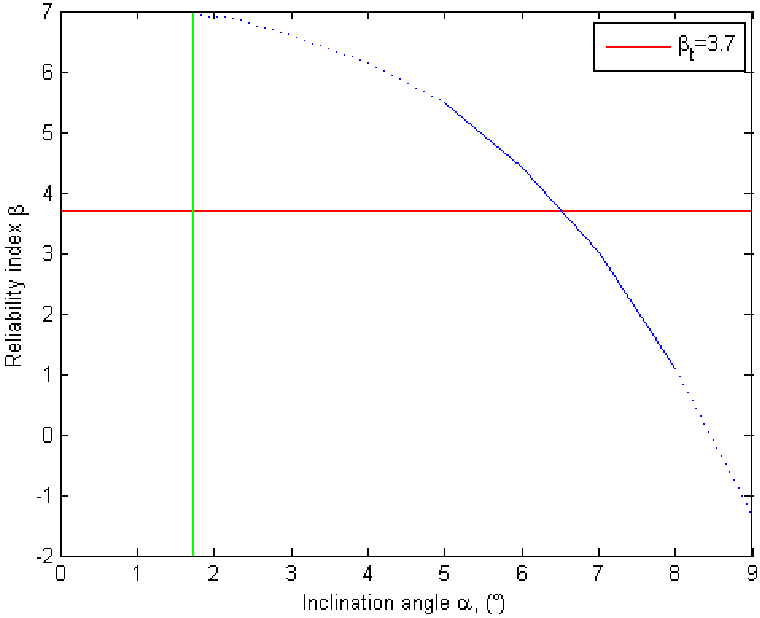

Figure 7.11: The reliability index β as a function of the inclination angle α for wind action ($t_r = 50$ years).

been evaluated considering a natural period $T_1 = 1.67\ s$ (Table 7.1), detected from the global analysis on the damaged model. As it is easy to understand intuitively, in case of seismic action the reliability index decreases faster than considering only the wind force. For $\alpha < 3°$ the structure can be considered reliable, but once that threshold is exceeded, the structural element is not safe anymore (Fig. 7.13).

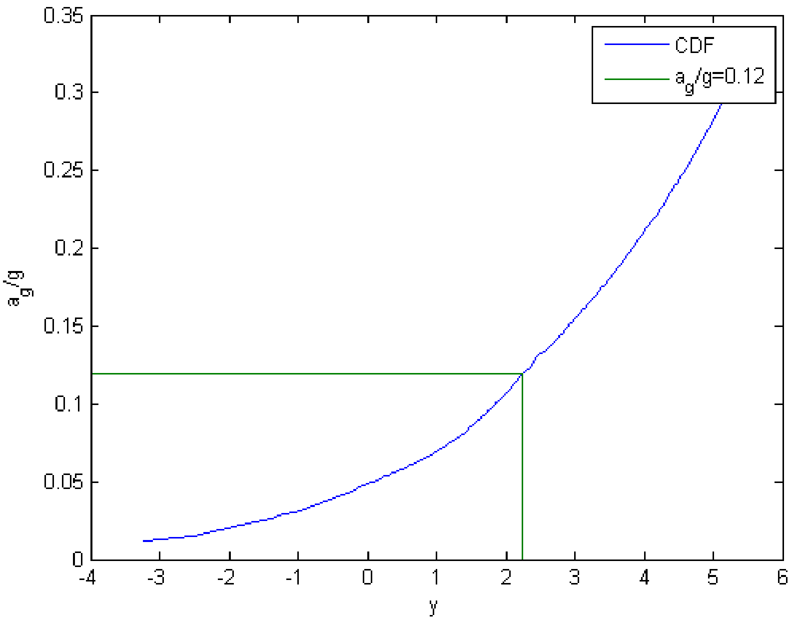

Figure 7.12: CDF of the ratio (a_g/g) $(t_r = 50$ years).

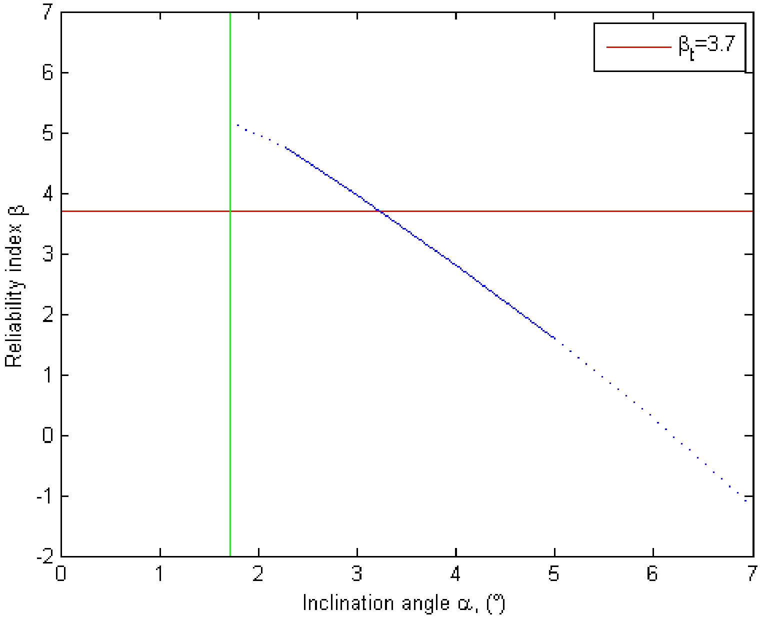

Figure 7.13: The reliability index β as a function of the inclination angle α for seismic action ($t_r = 50$ years).

7.4 Summary

Probabilistic methods and Bayesian updating techniques are often invoked in order to carefully quantify the reliability of existing structures. However, applying these procedures to ancient buildings presents several problems. First of all, the definition of probability models for the variables involved in the assessment, and especially material properties, is a difficult task. Historic buildings are designed according to empirical criteria and without any formal approach, therefore there is a lack of information from the design stage. Furthermore, the nature of the material commonly implemented, namely masonry and wood, are heterogeneous and non-isotropic, and the definition of appropriate probability models for their relevant mechanical properties is affected by great uncertainty. Finally, destructive tests may entail a loss of the cultural value embodied in the building and should be avoided. These factors complicate the determination of prior probability models and the calibration with classical Bayesian techniques. A further hurdle is represented by the fact that ancient buildings have usually been subjected to several alterations, that are difficult to recognize and may result in greater model uncertainty and unpredictable failure modes. If these issues are not considered, any probabilistic approach to the safety of historic structures will be highly untrustworthy and the potential benefits of implementing such procedures vanished. In order to properly address the above-mentioned issues, a methodology for performing a probabilistic reliability assessment of historic structures has been proposed. The key steps of the procedure are the following:

1. the definition of the building history through extensive investigation and documentary search, in order to locate possible alteration;

2. a global analysis, in order to identify damage scenarios and understand causes of degradation;

3. a local study, where the reliability of the structure is evaluated;

4. the updating of prior probability models when new evidence becomes available.

The opinion of historical buildings' experts fulfils a special role in the proposed methodology, not only for establishing prior probability models and identifying most relevant failure modes but also in order to confirm the obtained results. The proposed method has been then applied to a relevant case study, an ancient masonry aqueduct affected by settlements and out-of-plane overturning, whose safety against horizontal actions is highly questionable. We have seen that a probabilistic approach allows for a rational treatment of the uncertainty that affects the assessment process and for an easy evaluation of their influence on the computed result. In absence of any empirical data about the structure, Bayesian updating techniques have been applied in an often disregarded way, namely to refine the probability model of environmental loads, in order to obtain a more sound *pdf* of the wind speed to implement in the analysis. The study is aimed at defining the limit inclination for which the structure is not safe anymore. Of course, when the pillars are vertical or their inclination is low ($1° - 2°$) the high values obtained for the reliability index β simply indicate that the structure is practically not sensitive to horizontal actions. But increasing the inclination, the sensitivity considerably increases. Since the reliability index depends on the assumption, it represents a rough figure of the structural safety. However, in comparative studies, the index serves as a guide for understanding under which circumstances the structure might be in danger. In the present case, the results confirm the outcomes of deterministic analysis previously carried out, according which the structure is safe when it stands in a vertical position while it is not safe anymore when the inclination becomes greater than $3°$ if subjected to seismic action and $6°$ in case of wind force. In conclusion, the proposed methodology could represent a benchmark guideline for the study of historical structures, the collection of relevant information and their logical use; a probabilistic approach could be a valuable tool, additional to standard assessment procedure, for evaluating the safety of historic buildings.

Chapter 8

Conclusion and outlook

In this chapter the conclusion about the work are drawn. The main goals achieved are summarised and suggestions for the future steps of the research are given.

Contents

8.1 Conclusion

The main goals achieved during the development of this work are summarised in the following.

Development of a methodology for the probabilistic reliability assessment of heritage buildings. The procedure has been developed recognizing at first the main characteristics of those structures:

- They are designed according to empirical bases and architectural canons.
- Data regarding material mechanical parameters is difficult to collect, especially because destructive investigation cannot be applied and non-destructive techniques are frustrated by the heterogeneity of the material.
- Transfer functions are affected by great uncertainty because the assumptions behind classical analysis methods are scarcely respected; in particular, model uncertainty is very high in presence of structural alteration.

The procedure thus proposes:

- To use FEM analysis in order to explain the state of damage and to prove the failure scenarios identified during the early investigation.
- To develop *ad hoc* local models, able to capture particular resistance mechanisms and express in a simple way the performance function, so that the probability of failure can be computed applying simple approximation methods.
- To refine the probability models applying non-destructive approaches, also considering the updating of action and action effects.

A practical application of the procedure to a relevant case study has been also developed. It is shown how more sound reliability estimation can be obtained by updating the wind speed with the results of field study.

Development of a methodology for the reliability assessment of complex structures based on the Bayesian updating of the finite element model. The procedure is aimed at solving a problem often encountered in the reliability assessment of complex structures, whose analysis requires the definition of a FEM: the evaluation of failure probability with implicit limit states. The procedure proposes to explicit the limit state by reducing the uncertainty in the input random variables of the FEM to acceptable levels. The updating is carried out applying Bayes' theorem and collecting measurement of the system response. It is especially suggested to define a surrogate model of the output that can be measured, according to which it will be possible to update the prior distribution without simulating the model anymore. A practical application of this procedure to a relevant case study is also developed. The surrogate model has been created according to the gPCE theory: it is shown how this approach allows the updating of the input random variables considering the results of static and dynamic tests simultaneously. Based on the author's knowledge the way in which the updating has been performed is very promising and original, in addition to the fact that is a completely non-destructive approach.

Development of a methodology for the identification of material classes and their statistical parameters analysing secondary experimental test data. The procedure is aimed at solving a problem often encountered when secondary databases of test results are available, namely that individual results are not associated to any material population, so that their statistical analysis is a complicated process. According to the proposed procedure, the problem can be solved by performing a cluster analysis directly on the data. The cluster analysis is based on Gaussian mixture models, therefore it can be performed any time the uncertainty regarding the mechanical parameter can be expressed through a Gaussian distribution. The procedure, that represents a completely new approach to the problem, succeeds in extrapolating valuable information for the reliability assessment of existing structures from secondary databases of test results.

Identification of concrete resistance classes during the 1960s and their statistical parameters. The proposed methodology has been applied in order to investigate over concrete resistance classes in Italy during the 1960s. A database of results of compressive tests on standard cubes carried out during the 1960s has been considered. Results reveal that approximately six concrete classes can be identified, characterised by a standard deviation of about 4.5-5 MPa, practically independent from the strength, and a coefficient of variation that ranges from 0.14 to 0.07.

Development of a methodology according which prior distribution for material mechanical parameters are defined based on past data. The methodology previously described also allows to easily assess the uncertainty affecting material mechanical parameters, that can be further updated applying Bayes' theorem at a later date. It is thus possible to define more sound prior distributions based on the analysis of past data, as suggested by the empirical Bayes approach, rather than engineer's degree of belief. This approach is particularly appropriate since the statistical parameters can be here interpreted as real random variables.

8.2 Outlook

During the writing and the development of this thesis several issues emerged which deserve to be deepened by future researches. Some of these are listed in the following.

Use of reliability methods for planning structural interventions Reliability methods confirm themselves as very useful procedures for comparison purposes. For this reason, their use is suggested not only for drawing fragility estimates (as they have been implemented in this work) but also for carefully planning future interventions on the structures. A cautious plan of strengthening intervention is especially important in case of historical buildings, since similarly to destructive investigations they usually lead to a loss of the cultural value embodied in the building. It is therefore of paramount importance to limit the interventions to those strictly necessary. Reliability methods allow not only the definition of appropriate safety targets but also the minimum intervention according which that target can be achieved.

Use of random fields for the description of spatial variability of random properties In case of complex existing buildings, characterized by spatially heterogeneous material and whose behaviour can be investigated only through numerical simulation, advanced methods should be applied, that are able to capture the spatial variability of the material mechanical properties. It is possible to resort to the gPCE theory not only for facilitating the updating of the input parameters given measurements of the structural global response but also for the definition of the random fields describing the stochastic process; also the random fields defined over a FEM of the structure can be updated. The applicability of random fields for serving the above-mentioned purpose will be explored in the future.

Use of Bayesian Networks and other artificial intelligence techniques for supporting decision making tasks in the reliability assessment of existing structures Civil engineering invokes an increasing need for computer applications that show intelligent behaviour for supporting decision making tasks. This ability depends on how well the knowledge connected to that field is represented and processed, or in other words, how well the knowledge representation problem is solved [98]. A promising approach that can handle vague and uncertain knowledge is represented by Bayesian Networks. It is thus interesting to notice that fitting a Gaussian mixture model leads automatically, by its own nature, to the definition of a Bayesian Network. In this case, the network is very basic and comprises only two nodes: one node is represented by the material class, while the other node is represented by the material strength. The first node has many states as the number of identified class or population; at each node, a discrete probability (the so-called probability components, already mentioned in Chapter 5) is attached; the second node is indeed characterized by continuous distributions. Bayesian Networks have already been successfully applied in civil engineering for the definition of material mechanical characteristics [40, 150]. However, none of this research has explored the applicability of this technique for assessing material strength in existing structures. In order to make the Bayesian Network serving this purpose, further steps are required: a simple idea is to improve the network with additional variables representing the results of non-destructive techniques of structural investigation. By defining the general dependencies of the non-destructive outcome on the concrete class, it is possible to draw inference on the concrete class and thus on the concrete compressive strength considering the results of non-destructive techniques carried out on the structure. This method can be used in order to identify homogeneous concrete with respect to some mechanical properties, but also to define probability models which take into account the uncertainty in the definition of the class. The future steps of this research will follow this direction.

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